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Degeneracies in the theory of plane harmonic wave propagation in anisotropic heat-conducting elastic media

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Contents

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This paper explores the unusual hierarchy of degeneracies in the linear theory of thermoelasticity. In classical elastic wave theory all degeneracies take the form of acoustic axes, that is directions in which two or all three plane bulk waves have equal speeds. In dynamical thermoelasticity four plane harmonic waves can travel in an arbitrary direction, and there are two types of degeneracy. The first type arises when two or more waves have equal slownesses, normally complex, and the second type when the coefficient matrix of the governing system of differential equations has a repeated zero eigenvalue. Each type of degeneracy is of two possible kinds, so the number of cases in which at least one degeneracy occurs is eight. It is shown that only three of these possibilities can actually exist and in only one of them are both types of degeneracy present. The effects of thermomechanical interaction on the modes of wave propagation are then minimal. An analysis of the degeneracies, their interconnexion and their influence on the nature of thermoelastic waves occupies the first part of the paper. In the second part the relationship of classical elastodynamics to linear thermoelasticity is studied, in respect of degeneracy, by considering small thermoelastic perturbations of an acoustic axis. The underlying degeneracy is either removed by the perturbation or divided into one or two pairs of thermoelastic degeneracies. The directions in which the new degeneracies appear are determined and the properties of the associated degenerate waves discussed in detail.

1. Introduction

The classical theory of elastic wave propagation embodies an approximation additional to linearization: the ability of the transmitting material to conduct heat is ignored. The linear theory of thermoelasticity is free from this restriction and therefore provides a framework within which a rational assessment can be made of the inherent physical limitation of classical elastodynamics. Linear thermoelasticity has a long history, indications of which can be found in articles by Chadwick (1960) and Carlson (1972). But formulations with a sound thermomechanical basis, general enough to underlie the theory of elastic wave propagation in media which are anisotropic and possibly pre-stressed, are comparatively modern developments (Green 1962; Chadwick 1979). The treatment of linear dynamical thermoelasticity in the second of these references includes a discussion of plane harmonic wave propagation which has been greatly extended in recent years by Scott (1989, 1993). In particular, detailed studies have been made of the stability of the possible modes of wave propagation and the kinds of degeneracy to which the modes are subject.

In classical elastodynamics three homogeneous plane waves can propagate in the direction of an arbitrary unit vector $\boldsymbol{n}$ (the wave normal). Degeneracy occurs when the phase speeds of two or all three waves are equal and $\boldsymbol{n}$ is then said to define an acoustic axis. The equality of two wave speeds is a necessary and sufficient condition for the corresponding waves to combine as a circularly polarized wave (Boulanger & Hayes 1993, § 10.2). The analogous notion in dynamical thermoelasticity is equality of two roots of the secular equation, determining the slownesses of the four modes of wave propagation with wave normal $\boldsymbol{n}$. These modal roots are generally complex, implying that the modes are dispersive with frequency-dependent damping. It has been found by Scott (1993) that the conditions for a degenerate modal root and for a circularly polarized mode no longer coincide, and it is this difference which has prompted the present paper. In Scott's work interest centres on an isotropic null vector of the coefficient matrix of the governing system of differential equations (an isotropic vector being a complex vector with zero modulus, representing the amplitude of a circularly polarized wave). The approach followed here focuses on the zero eigenvalue of this matrix which, when doubly degenerate (i.e. of multiplicity 2), has an isotropic associated eigenvector. Since the complex coefficient matrix is non-Hermitian, a degeneracy of the zero eigenvalue can be of two types, semisimple or non-semisimple, according as a complete set of eigenvectors does or does not exist (see, for example, Pease 1965, ch. III).

The first part of the paper (§§ 2–6) is principally concerned with the analysis of the degeneracies mentioned above, the relations between them and their impact on the character of the modes to which they relate. In § 2 an outline of the theory of thermoelastic wave propagation is presented, based largely on the papers of Chadwick (1979) and Scott (1989, 1993), but broadened to cover the possibility of one or more modal roots being degenerate. The subsequent investigation of degeneracies involves generalized transverse and longitudinal waves. These are special types of plane harmonic thermoelastic waves with highly simplified properties, introduced by Chadwick (1979) as an extension of ideas advanced earlier in connexion with the theory of acceleration waves in heat-conducting elastic materials (Chadwick & Currie 1974*a*, *b*). Section 3 is devoted to an account of these waves which incorporates a diversity of new results. Degeneracy of a modal root is considered in § 4 and degeneracy of the zero eigenvalue of the coefficient matrix in § 5. It is proved in § 4 that a doubly degenerate modal root may be associated with a composite mode the amplitude of which varies linearly with distance in the direction of travel. In § 5 semisimple eigenvalue degeneracy is found to have a strongly pronounced effect on all the modes of thermoelastic wave propagation. The wave normal for which the degeneracy occurs is independent of frequency and two of the modes are generalized longitudinal, the others generalized transverse. Relations between the degeneracies are established in § 6. It is concluded that modal-root and eigenvalue degeneracies coexist if and only if the latter is semisimple, in which case the the simplifications specified in § 5 ensue. To conclude § 6 an example is given of a degenerate modal root unaccompanied by eigenvalue degeneracy. The modes in question are generalized longitudinal and the solution supplements an earlier investigation of this coupling by Chadwick (1973).

In the second part (§§ 7 and 8) we turn from exact to approximate analysis. It has been shown by Chadwick (1979) that a thermoelastic perturbation of classical elastodynamics can be appropriately constructed in terms of a dimensionless frequency χ which is very small in practically every physically realizable elastic wave motion. The unperturbed state consists of the zero-frequency limit $\chi \to 0$ and a selected wave normal $\boldsymbol{n}_0$. The limit $\chi \to 0$ is identical with the limit of zero thermal conductivity, so the unperturbed modes duly conform to the equations of classical elastodynamics. In §§ 7 and 8, $\boldsymbol{n}_0$ defines an acoustic axis, a situation which, by implication, is excluded from Chadwick's perturbation procedure. The aim in § 7 is to locate wave normals close to $\boldsymbol{n}_0$ for which the two modes of thermoelastic wave propagation with equal speeds in the unperturbed state are degenerate for small χ. The solution of this problem depends on a classification of acoustic axes in classical elastodynamics originated by Alshits *et al.* (1985). Remarkably, the theory providing the basic conditions for the classification also determines the degenerate perturbed wave normals. It transpires that, as χ increases from zero, the direction of degeneracy vanishes or divides into one or two pairs of degenerate normals. In each of these directions a composite degenerate mode propagates, the amplitude varying linearly with the distance travelled. The displacement and the temperature change produced by such a composite mode are derived in § 8, in the leading approximation, and the rather complicated evolution of the mode is described.

2. Basic theory of thermoelastic waves

(a) Governing equations

A body $\mathcal{B}$, composed of a heat-conducting elastic material, is assumed to possess an equilibrium configuration B_e in which the density, temperature and Cauchy stress have the uniform values $\bar{\rho}$, $\bar{T}$, $\bar{\boldsymbol{\sigma}}$, respectively, and the heat flux is everywhere zero. The instantaneous isothermal and isentropic elastic moduli, the temperature coefficient of stress, the symmetric part of the thermal conductivity and the specific heat at constant deformation, all evaluated in B_e, are in turn denoted by $\tilde{\boldsymbol{B}}$, $\hat{\boldsymbol{B}}$, $\boldsymbol{\beta}$, $\boldsymbol{k}$ and c. In relation to an arbitrary orthonormal basis e, the tensorial quantities have components $\tilde{B}_{ijkl}$, $\hat{B}_{ijkl}$, β_{ij} and k_{ij}. We note the connexion

$$\hat{B}_{ijkl} = \tilde{B}_{ijkl} + (\bar{\rho}c/\bar{T})^{-1}\beta_{ij}\beta_{kl} \tag{2.1}$$

and the symmetry properties

$$\tilde{B}_{klij} = \tilde{B}_{ijkl}, \qquad \hat{B}_{klij} = \hat{B}_{ijkl}, \qquad \beta_{ji} = \beta_{ij} \tag{2.2}$$

(Chadwick 1979, p. 196). Where no statement is made to the contrary, vector and tensor components are in future referred to e.

In the absence of external body forces and heat supply, the components of displacement u_i and the temperature change θ in a thermomechanical process in which $\mathcal{B}$ makes only small departures from B_e are governed by the partial differential equations

$$\tilde{B}_{pirs}u_{s,pr} - \beta_{ip}\theta_{,p} = \bar{\rho}\ddot{u}_i, \qquad k_{pq}\theta_{,pq} - \bar{T}\beta_{pq}\dot{u}_{p,q} = \bar{\rho}c\dot{\theta} \tag{2.3}$$

(Chadwick 1979, eqns (19)). Here the dot and comma notations are used for differentiation with respect to the time t and Cartesian coordinates x_i associated with e, and summation applies on repeated subscripts. The change of entropy produced by the process is

$$\eta = \bar{\rho}^{-1}\beta_{pq}u_{p,q} + \bar{T}^{-1}c\theta. \tag{2.4}$$

(b) Modes of plane harmonic wave propagation

We assume that the temporal and spatial dependences of u_i and θ are through the exponential factor $\exp(-\mathrm{i}\omega t)$ and the distance $x = x_p n_p$, ω being a positive real angular frequency and n_i the components of a real unit vector $\boldsymbol{n}$. Then (2.3) reduce to the ordinary differential equations,

$$\left.\begin{aligned}(\tilde{B}_{pirs}n_p n_r d_x^2 + \bar{\rho}\omega^2\delta_{is})u_s - \beta_{ip}n_p d_x\theta = 0,\\ \mathrm{i}\beta_{pq}n_p d_x u_p + \{(\omega\bar{T})^{-1}k_{pq}n_p n_q d_x^2 + \mathrm{i}\bar{\rho}c\bar{T}^{-1}\}\theta = 0,\end{aligned}\right\} \tag{2.5}$$

in which $d_x = \mathrm{d}/\mathrm{d}x$ and δ is the Kronecker delta. Following Chadwick (1979, p. 200) and Scott (1993, § 4(a)), we define the dimensionless quantities

$$\left.\begin{aligned} \tilde{Q}_{ij} &= \gamma^{-1}\tilde{B}_{pirj}n_p n_r, & \hat{Q}_{ij} &= \gamma^{-1}\hat{B}_{pirj}n_p n_r, \\ b_i &= (\bar{\rho}c\gamma/\bar{T})^{-1/2}\beta_{ip}n_p, & \chi &= (\gamma c)^{-1}\omega k_{pq}n_p n_q, \end{aligned}\right\} \tag{2.6}$$

and the scaled temperature change

$$\phi = -\mathrm{i}\omega^{-1}(c/\bar{T})^{1/2}\theta, \tag{2.7}$$

having the physical dimension of length. In equations (2.6), γ is a typical stress and the dependence of $\tilde{Q}_{ij}$, $\hat{Q}_{ij}$, b_i and χ on $\boldsymbol{n}$ is tacit. Alternative definitions of γ have been suggested by Chadwick (1979, eqn (25)) and Scott (1989, eqn (21)). Equations (2.5) can now be rewritten in the matrix form

$$L(d_x)v = 0, \tag{2.8}$$

where

$$L(d_x) = \begin{bmatrix} \tilde{Q}d_x^2 + \zeta^2 I & -\mathrm{i}\zeta b d_x \\ -\mathrm{i}\zeta b^{\mathrm{T}} d_x & -\mathrm{i}\chi d_x^2 + \zeta^2 \end{bmatrix} \tag{2.9}$$

and

$$v = (u^{\mathrm{T}}, \phi)^{\mathrm{T}}, \qquad \zeta = (\bar{\rho}\omega^2/\gamma)^{1/2}.$$

An obvious extension of the previous notation has been brought into use in equations (2.8) and (2.9): $\tilde{Q} = [\tilde{Q}_{ij}]$ is the 3×3 matrix of components of $\tilde{\boldsymbol{Q}}$, $I = [\delta_{ij}]$ is the 3×3 identity matrix, $u = [u_i]^{\mathrm{T}}$ and $b = [b_i]^{\mathrm{T}}$ are the 3×1 column vectors of components of $\boldsymbol{u}$ and $\boldsymbol{b}$, and the superscript T denotes transposition. Consistent use of matrix notation is made up to the end of § 6 and in § 8, but we continue to refer to the wave normal as $\boldsymbol{n}$ to avoid confusion with the index n appearing first in equation (2.10) below.

Through the introduction of additional dependent variables $u' = d_x u$ and $\phi' = d_x\phi$, equation (2.8) can be transformed into the first-order system $d_x y = Ay$ where $y = [u^{\mathrm{T}}, \phi, u'^{\mathrm{T}}, \phi']^{\mathrm{T}}$ and A is an 8×8 complex matrix. From the general solution of the latter system (as given, for instance, by Miller (1964, ch. 8)) and the identity

$$\det(A - \mathrm{i}\omega s I_8) = \mathrm{i}\chi^{-1}(\det\tilde{Q})^{-1}\det L(\mathrm{i}\omega s),$$

in which I_8 is the 8×8 identity matrix, we deduce that the general solution of (2.8) is of the form,

$$v = \sum_{k=1}^{n} \exp\{\mathrm{i}\omega(s_k x - t)\} \sum_{l=1}^{m_k} x^{l-1} V_k^{(l)}, \tag{2.10}$$

where s_k, $k = 1, \ldots, n$, are the distinct roots of

$$\det L(\mathrm{i}\omega s) = 0 \tag{2.11}$$

and m_k is the multiplicity of the root s_k. In (2.10), $n \leqslant 4$ since $\det L(\mathrm{i}\omega s)$ is a polynomial of degree 4 in s (see (2.9)).

Equation (2.10) represents a linear combination of n plane harmonic waves, travelling with common angular frequency ω and slownesses $s_1, \ldots, s_n$ (generally complex) in the direction of the wave normal $\boldsymbol{n}$. The individual waves are solutions of the governing equations (2.3) and are referred to as *modes* of thermoelastic wave propagation. When $n = 4$, $m_k = 1$ for $k = 1, \ldots, 4$.

It follows from (2.9) that

$$L(\mathrm{i}\omega s) = \zeta^2 w^{-1} S(w),$$

where

$$S(w) = \begin{bmatrix} wI - \tilde{Q} & w^{1/2} b \\ w^{1/2} b^{\mathrm{T}} & w + \mathrm{i}\chi \end{bmatrix} \tag{2.12}$$

and the further dimensionless quantity w is defined by

$$w = \bar{\rho} s^{-2}/\gamma = (\omega s/\zeta)^{-2}. \tag{2.13}$$

The *secular equation* (2.11), determining the slownesses of the modes, is therefore equivalent to

$$\det S(w) = 0. \tag{2.14}$$

The tensors $\gamma\tilde{\boldsymbol{Q}}$ and $\gamma\hat{\boldsymbol{Q}}$, with components defined by equations $(2.6)_{1,2}$, are, respectively, the isothermal and isentropic acoustical tensors associated with the wave normal $\boldsymbol{n}$: $\tilde{\boldsymbol{Q}}$ and $\hat{\boldsymbol{Q}}$ are symmetric by virtue of $(2.2)_{1,2}$. The connexion

$$\hat{Q} = \tilde{Q} + bb^{\mathrm{T}} \tag{2.15}$$

between the corresponding matrices of components follows from equations (2.1) and $(2.6)_{1,2,3}$. With the aid of equations $(\mathrm{A}\,1)_{1,2}$, (A 9) and $(\mathrm{A}\,10)_{1,2}$ in the appendix, the secular equation (2.14) can be expressed in the alternative forms,

$$(w + \mathrm{i}\chi)\det(wI - \tilde{Q}) - wb^{\mathrm{T}}(wI - \tilde{Q})^{*} b = 0 \tag{2.16}$$

(Scott 1993, eqn (4.11)) and

$$\mathrm{i}\chi \det(wI - \tilde{Q}) + w \det(wI - \hat{Q}) = 0 \tag{2.17}$$

(Chadwick 1979, eqn (32)).

It is obvious from (2.16) that the secular equation is a polynomial equation of degree 4 in w. The roots w_k, $k = 1, \ldots, n$, are called the *modal roots* and the allied slownesses are

$$s_k = (\gamma w_k/\bar{\rho})^{-1/2}. \tag{2.18}$$

A modal root is said to be *simple* when its multiplicity, m_k, is 1 and *degenerate* when $m_k > 1$. For convenience these terms are also applied to the mode itself.

Since $S(w)$ is a complex matrix, the modal roots are normally complex-valued functions of the (real, non-negative) dimensionless frequency χ, depending also on the wave normal $\boldsymbol{n}$. The slownesses are likewise complex, in view of (2.18), and it follows from (2.10) that the modes are subject to damping, or growth, and dispersion. The possibility of growth is excluded, rendering the modes linearly stable, if $\operatorname{Im} w_k \leqslant 0$ for $k = 1, \ldots, n$ and all $\chi \geqslant 0$ (Scott 1993, eqn (4.34)). It has been proved by Scott (1989, § 3) that all the modes with wave normal $\boldsymbol{n}$ are linearly stable if $\tilde{\boldsymbol{Q}}$ and $\boldsymbol{k}$ are positive definite and $c > 0$. We assume that these conditions are satisfied for each wave normal considered, and also that $\boldsymbol{\beta}$ is non-singular (cf. Chadwick 1979, p. 197). On account of equation (2.15), $\hat{\boldsymbol{Q}}$ is also positive definite.

(*c*) *Simple modes*

The displacement and scaled temperature change produced by a mode of thermoelastic wave propagation with simple modal root w_k are given by equations (2.10) and (2.18) as

$$v_k = \exp[\mathrm{i}\omega\{(\gamma w_k/\bar{\rho})^{-1/2}x - t\}]V_k, \tag{2.19}$$

the superscript (1) being omitted from the constant amplitude V_k. Entering this wave form into equation (2.8) and making use of (2.9), (2.12) and (2.13), we confirm that

$$S(w_k)V_k = 0. \tag{2.20}$$

Since

$$\det S(w_k) = 0, \tag{2.21}$$

equations (A 3) and (A 9) show the general solution of (2.20) to be $V_k = [U_k^{\mathrm{T}}, \varPhi_k]^{\mathrm{T}}$ where

$$U_k = -aw_k^{1/2}(w_kI - \tilde{Q})^*b, \qquad \varPhi_k = a\det(w_kI - \tilde{Q}), \tag{2.22}$$

and a is an arbitrary scalar. The solution (2.22) is vacuous when w_k is an eigenvalue of $\tilde{Q}$ with associated eigenvector orthogonal to b. We deal with this exceptional case in § 3 *a*. Invoking the version (2.16) of the secular equation, we deduce from (2.22) the *temperature relation*,

$$\varPhi_k = -w_k^{1/2}(w_k + \mathrm{i}\chi)^{-1}b^{\mathrm{T}}U_k, \tag{2.23}$$

linking the displacement amplitude U_k and the scaled temperature change amplitude $\varPhi_k$, and the *propagation condition*,

$$T(w_k)U_k = 0, \tag{2.24}$$

wherein

$$\begin{aligned} T(w) &= wI - \tilde{Q} - w(w+\mathrm{i}\chi)^{-1}bb^{\mathrm{T}} = wI - \hat{Q} + \mathrm{i}\chi(w+\mathrm{i}\chi)^{-1}bb^{\mathrm{T}} \\ &= (w+\mathrm{i}\chi)^{-1}\{\mathrm{i}\chi(wI - \tilde{Q}) + w(wI - \hat{Q})\} \end{aligned} \tag{2.25}$$

(Chadwick 1979, eqns (29), (30), (34); Scott 1993, eqn (4.26)). In connexion with equations (2.23) and (2.25), it should be noted that $\mathrm{Re}\, w_k \neq 0$. For if w_k were imaginary it could not be an eigenvalue of the real symmetric matrix

$$\tilde{Q} + w_k(w_k + \mathrm{i}\chi)^{-1}bb^{\mathrm{T}}.$$

In particular,

$$w_k \neq -\mathrm{i}\chi \qquad \forall \chi > 0. \tag{2.26}$$

Equations (2.23) and (2.24) effectively separate the mechanical and thermal variables u_k and ϕ_k for a simple mode, and equations $(\mathrm{A}\,1)_3$ and (A 9) deliver

$$\det T(w) = 0 \tag{2.27}$$

as a further alternative to the secular equation (2.14). Although $T(w)$ is a 3×3 matrix, (2.27) is a polynomial equation of degree 4. According to (2.24), a null vector of $T(w_k)$ corresponding to a simple root w_k supplies the displacement amplitude, and the amplitude of the scaled temperature change follows from (2.23). The accompanying change in entropy is given by equations (2.4), $(2.6)_3$, (2.7), (2.19) and (2.23) as

$$\eta_k = \omega\chi w_k^{-1}(c/\bar{T})^{1/2}\varPhi_k \exp[\mathrm{i}\omega\{(\gamma w_k/\bar{\rho})^{-1/2}x - t\}]. \tag{2.28}$$

A simple mode of thermoelastic wave propagation which gives rise to no change of temperature thus travels without change of entropy. Moreover, by (2.23), ϕ_k and η_k vanish identically for all $\chi \geqslant 0$ if and only if

$$b^{\mathrm{T}}U_k = 0. \tag{2.29}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Simple modes with the property (2.29) are discussed in detail in § 3 *a*.

(*d*) *The low- and high-frequency limits*

The extremes of the frequency range are of crucial significance (Chadwick 1979, pp. 203–206).

In the low-frequency limit $\chi \to 0$, it is evident from equation (2.17) that the modal roots are the (positive, real) eigenvalues of the dimensionless isentropic acoustical tensor $\hat{Q}$ and zero, the latter (necessarily a simple root) corresponding to a diffusive mode. The other modes are undamped, non-dispersive waves. In the case of a simple mode, equations (2.24) and $(2.25)_2$ show that the displacement amplitude U_k is an eigenvector of $\hat{Q}$ associated with the modal root w_k and, from (2.23), the amplitude of the scaled temperature change is $\Phi_k = w_k^{-1/2} b^{\mathrm{T}} U_k$. In accord with the central position of $\hat{Q}$, equation (2.28) confirms that the mode propagates without change of entropy.

It follows from the definition $(2.6)_4$ that the low-frequency limit is realized when the thermal conductivity $\boldsymbol{k}$ is zero. This limit, and the isentropic conditions attending it, are therefore appropriate to the linearized dynamics to a non-heat-conducting elastic body, in short to classical elastodynamics. For most solid materials at non-cryogenic temperatures, the characteristic frequency $\omega^* = \gamma c/(k_{pq} n_p n_q)$ which divides ω in $(2.6)_4$ is at least of order 10^{12} Hz (see, for example, Chadwick 1960, table 1). The strong inequality $\chi(= \omega/\omega^*) \ll 1$ therefore applies in practically all physically realistic situations and it is used as a basis for approximation in §§ 7 and 8. Despite this limitation and the intervention of cut-offs imposed by the microstructure of the material when $\chi \gg 1$, a complete theoretical understanding of the modes of thermoelastic wave propagation requires an examination of the whole of the frequency range $\chi \geqslant 0$.

In the high-frequency limit $\chi \to \infty$, equation (2.17) implies that the modal roots are the (positive, real) eigenvalues of the dimensionless isothermal acoustical tensor $\tilde{Q}$. Each of the corresponding modes is an undamped, non-dispersive wave. The displacement amplitude of a simple mode is seen from equations (2.24) and $(2.25)_1$ to be an eigenvector of $\tilde{Q}$ associated with the modal root and, consistently with the governing role of $\tilde{Q}$, equation (2.23) affirms that the mode produces no change of temperature.

In the case in which there are four simple modes with wave normal $\boldsymbol{n}$, the first two terms of convergent expansions of the modal roots in direct and inverse powers of χ have been determined by Chadwick (1979, pp. 204–206) and used to obtain low- and high-frequency approximations to the speed of propagation and the attenuation coefficient of each mode.

3. Generalized transverse and longitudinal modes

(*a*) *Generalized transverse modes*

A mode of thermoelastic wave propagation with wave normal $\boldsymbol{n}$ and dimensionless frequency χ is said to be *generalized transverse* (GT) when its displacement amplitude U_k satisfies equation (2.29), that is when U_k is orthogonal to the vector b (which, by $(2.6)_3$, is non-zero, $\boldsymbol{\beta}$ being non-singular). It is plain from equations (2.24) and $(2.25)_{1,2}$ that if (2.29) holds for some $\chi > 0$, it applies for all $\chi \geqslant 0$. Moreover, U_k is an eigenvector of both $\tilde{Q}$ and $\hat{Q}$, and since these matrices are symmetric, real and independent of χ, the last two properties are inherited by U_k and also by the modal root w_k, which is a simultaneous eigenvalue of $\tilde{Q}$ and $\hat{Q}$. A GT mode is therefore undamped and non-dispersive: it is totally unaffected by thermomechanical interaction in the body $\mathcal{B}$.

If, for some $\boldsymbol{n}$, b is orthogonal to an eigenvector $\tilde{q}_k$ of $\tilde{Q}$, with associated eigenvalue $\tilde{w}_k$, then, from (2.15) and $(2.25)_1$, $\tilde{q}_k$ is an eigenvector of $\hat{Q}$, with accompanying eigenvalue $\tilde{w}_k$, and a null vector of $T(\tilde{w}_k)$ for all $\chi \geqslant 0$. As shown by Chadwick & Currie (1974*a*, Appendix), b is orthogonal to an eigenvector of $\tilde{Q}$ if and only if

$$[b, \tilde{Q}b, \tilde{Q}^2 b] = 0, \tag{3.1}$$

the square brackets denoting a scalar triple product. Thus (3.1) is a necessary and sufficient condition for the existence of a GT mode with wave normal $\boldsymbol{n}$, implicit in $\tilde{Q}$ and b. On the basis of this criterion, Chadwick & Currie (1974*a*, § 3) have given a topological argument which establishes the existence, in an arbitrary plane, of at least one wave normal for which, for all $\chi \geqslant 0$, one of the modes of thermoelastic wave propagation is GT.

Suppose that, for given values of $\boldsymbol{n}$ and χ, two modes have displacement amplitudes U_j, U_k and modal roots w_j, w_k. Then the propagation condition (2.24) yields the reciprocal relation

$$U_k^{\mathrm{T}} T(w_j) U_j - U_j^{\mathrm{T}} T(w_k) U_k = 0,$$

which, with the use of $(2.25)_1$, becomes

$$(w_j - w_k)\{U_j^{\mathrm{T}} U_k - \mathrm{i}\chi(w_j + \mathrm{i}\chi)^{-1}(w_k + \mathrm{i}\chi)^{-1}(b^{\mathrm{T}} U_j)(b^{\mathrm{T}} U_k)\} = 0. \tag{3.2}$$

Equation (3.2) has the following consequences.

GT 1. *If, for some* $\boldsymbol{n}$ *and* χ, *two modes have orthogonal displacement amplitudes and distinct modal roots, at least one of them is a GT mode.*

GT 2. *If, for some* $\boldsymbol{n}$, *one mode is a simple GT mode, then, for all* $\chi \geqslant 0$, *the plane orthogonal to the real displacement amplitude of this mode contains* b *and the displacement amplitudes of all the other modes.*

The result of combining equations (2.24) and $(2.25)_1$ and forming the scalar product of each term with $\bar{U}$, the complex conjugate of U, is

$$\bar{U}^{\mathrm{T}}(wI - \tilde{Q})U = w^{-1}(w + \mathrm{i}\chi)^{-1}(b^{\mathrm{T}} U)(b^{\mathrm{T}} \bar{U}). \tag{3.3}$$

If w is real, so is the left-hand side of (3.3) and, on taking imaginary parts, we find that $b^{\mathrm{T}} U = 0$. We have thus proved

GT 3. *If, for some* $\boldsymbol{n}$ *and* χ, *a mode of thermoelastic wave propagation has a real modal root, this mode is necessarily GT.*

It is apparent from GT 3 that the exceptional case, mentioned in § 2*c*, in which the general solution (2.22) fails is that in which w_k belongs to a GT mode. We could then replace (2.22) by $U_k = a\tilde{q}_k$, $\Phi_k = 0$, $\tilde{q}_k$ being a unit eigenvector of $\tilde{Q}$ associated with the eigenvalue w_k, or

$$U_k = -a w_k^{1/2}(w_k I - \tilde{Q})^* c, \qquad \Phi_k = 0,$$

where c is any vector such that $b \times c \neq 0$.

Lastly, for some $\boldsymbol{n}$, let the limiting values of a modal root w_k as $\chi \to 0$ and as

$\chi \to \infty$ be $\hat{w}_k$ and $\tilde{w}_k$, in turn. As noted in § 2 *d*, $\hat{w}_k$, $\tilde{w}_k$ are eigenvalues of $\hat{Q}$, $\tilde{Q}$, and therefore real. We conclude from GT 3 that

$$w_k = \hat{w}_k \quad \text{and} \quad w_k = \tilde{w}_k \tag{3.4}$$

are, individually, sufficient conditions for the mode in question to be GT. The conditions

$$\hat{Q} \quad \text{and} \quad \tilde{Q} \quad \text{have a common eigenvalue} \tag{3.5}$$

and

$$\det(w_k I - \hat{Q}) = \det(w_k I - \tilde{Q}) \tag{3.6}$$

also suffice for the existence of a GT mode. To justify (3.5) we observe that if $\hat{q}$, $\tilde{q}$ are eigenvectors of $\hat{Q}$, $\tilde{Q}$ associated with the common eigenvalue w, we have $\hat{Q}\hat{q} = w\hat{q}$, $\tilde{Q}\tilde{q} = w\tilde{q}$, whence, appealing to (2.15),

$$w\tilde{q}^{\mathrm{T}}\hat{q} = \tilde{q}^{\mathrm{T}}\hat{Q}\hat{q} = \tilde{q}^{\mathrm{T}}\tilde{Q}\hat{q} + (b^{\mathrm{T}}\hat{q})(b^{\mathrm{T}}\tilde{q}) = w\hat{q}^{\mathrm{T}}\tilde{q} + (b^{\mathrm{T}}\hat{q})(b^{\mathrm{T}}\tilde{q}).$$

Either $b^{\mathrm{T}}\hat{q} = 0$ or $b^{\mathrm{T}}\tilde{q} = 0$. If $\tilde{q}$ is orthogonal to b, $\tilde{q}$ is also orthogonal to $\tilde{Q}b$ and $\tilde{Q}^2 b$, and the condition (3.1) is satisfied. If $\hat{q}$ is orthogonal to b, the same conclusion is reached with the aid of (2.15). With regard to (3.6), we infer from (2.17) and (2.26) that w_k is an eigenvalue of both $\hat{Q}$ and $\tilde{Q}$, and then, from either of (3.4), that the mode with root w_k is GT.

(*b*) *Generalized longitudinal modes*

In keeping with the terminology introduced in § 3 *a*, we describe as *generalized longitudinal* (GL) a mode of thermoelastic wave propagation for which the displacement amplitude U_k is a scalar multiple of b. Equations (2.24) and $(2.25)_{1,2}$ imply that if U_k is a scalar multiple of b for some $\chi > 0$, the same is true for all $\chi \geqslant 0$. Thus, for a GL mode, U_k can be taken to be real and independent of χ. However, these simplifications do not extend to the modal root w_k allied to U_k: in consequence of GT 3, a GL mode is necessarily damped and dispersed.

Equations (2.24) and $(2.25)_{1,2}$ stipulate further that when $\boldsymbol{n}$ is the wave normal of a GL mode, b is an eigenvector of both $\hat{Q}$ and $\tilde{Q}$. Suppose that $\tilde{Q}$ has distinct eigenvalues $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{w}_3$ with associated unit eigenvectors $\tilde{q}_1 = (b^{\mathrm{T}}b)^{-1/2}b$, $\tilde{q}_2$, $\tilde{q}_3$. Then, inserting the spectral form of $\tilde{Q}$ into $(2.25)_1$,

$$T(w) = \{w - \tilde{w}_1 - w(w + \mathrm{i}\chi)^{-1}b^{\mathrm{T}}b\}(b^{\mathrm{T}}b)^{-1}bb^{\mathrm{T}} + (w - \tilde{w}_2)\tilde{q}_2\tilde{q}_2^{\mathrm{T}} + (w - \tilde{w}_3)\tilde{q}_3\tilde{q}_3^{\mathrm{T}}. \tag{3.7}$$

We deduce that

$$\det T(w) = \{w - \tilde{w}_1 - w(w + \mathrm{i}\chi)^{-1}b^{\mathrm{T}}b\}(w - \tilde{w}_2)(w - \tilde{w}_3),$$

and hence, from the secular equation (2.27), that the modal roots of the modes with wave normal $\boldsymbol{n}$ are $\tilde{w}_2$, $\tilde{w}_3$, both real, and the roots of the quadratic equation

$$w^2 - (\tilde{w}_1 + b^{\mathrm{T}}b - \mathrm{i}\chi)w - \mathrm{i}\chi\tilde{w}_1 = 0. \tag{3.8}$$

By GT 3, the modes with roots $\tilde{w}_2$, $\tilde{w}_3$ are GT. Their displacement amplitudes are scalar multiples of $\tilde{q}_2$, $\tilde{q}_3$, both orthogonal to b. For each of the other modes, we see from (2.24) and (3.7) that the displacement amplitude is a scalar multiple of b. We have thus proved

GL 1. *A wave normal which admits a GL mode and for which $\tilde{Q}$ has distinct eigenvalues also admits a second GL mode with the same displacement amplitude* *and two simple GT modes. Such a wave normal is henceforth referred to as a GL-normal.*

If, for some $\boldsymbol{n}$, b is an eigenvector of $\tilde{Q}$ with corresponding eigenvalue $\tilde{w}_1$, it is clear from equation $(2.25)_1$ that b is a null vector of $T(w_k)$ for all $\chi \geqslant 0$ when w_k is a root of (3.8). As seen above, $\boldsymbol{n}$ then defines a GL normal. Conversely, if $\boldsymbol{n}$ is aligned with a GL normal, two of the isonormal modes are GL. Accordingly, there exists w_k such that b is a null vector of $T(w_k)$ and, by $(2.25)_1$, an eigenvector of $\tilde{Q}$. These considerations establish

GL 2. *$\boldsymbol{n}$ defines a GL normal if and only if b is an eigenvector of $\tilde{Q}$.*

Reasoning by Chadwick & Currie (1974*b*, § 4), combining GL 2 with a topological theorem, proves that, regardless of the pre-stress and the material symmetry, there always exists at least one GL normal. By GL 1, a GL mode is accompanied by two GT modes. A GL normal thus marks the intersection on the unit sphere $\mathcal{U}$, representing the totality of wave normals, of two curves traced out by wave normals for which a GT mode exists. These curves are determined by equation (3.1) and, as noted in § 3 *a*, their union has at least one point of intersection with each plane through the centre of $\mathcal{U}$. When there is no pre-stress and the heat-conducting elastic material has symmetry, the normal to a plane of symmetry is also a GL normal. In this case the wave normal $\boldsymbol{n}$ is an eigenvector of $\boldsymbol{\beta}$, so the GL modes are longitudinal and the GT modes transverse in the conventional sense.

A detailed study of the implications of equation (3.8) has been made by Chadwick (1973). At both extremes of the frequency range one GL mode is wavelike and the other diffusive, but it is possible for a mode which is wavelike when $\chi \ll 1$ to be diffusive when $\chi \gg 1$ and vice versa.

Sufficient conditions for $\boldsymbol{n}$ to define a GL normal can also be specified in terms of modal properties.

GL 3. *If a wave normal $\boldsymbol{n}$ admits two simple GT modes, it is a GL normal.*

This is a corollary of GL 2: the displacement amplitudes of the GT modes are eigenvectors of $\tilde{Q}$, orthogonal to each other and to b. Hence b is an eigenvector of $\tilde{Q}$.

GL 4. *If, for some $\boldsymbol{n}$ and $\chi > 0$, the displacement amplitudes of two modes are scalar multiples of one another and $\tilde{Q}$ has distinct eigenvalues, $\boldsymbol{n}$ defines a GL normal.*

Let w_1 and w_2 be the modal roots and U a displacement amplitude common to both modes. Then equations (2.24) and $(2.25)_1$ give

$$(w_k I - \tilde{Q})U = w_k(w_k + \mathrm{i}\chi)^{-1}(b^{\mathrm{T}}U)b, \qquad k = 1, 2, \tag{3.9}$$

the difference of which is

$$(w_1 - w_2)U = \mathrm{i}\chi(w_1 + \mathrm{i}\chi)^{-1}(w_2 + \mathrm{i}\chi)^{-1}(b^{\mathrm{T}}U)b.$$

Since $w_1 \neq w_2$, U is a scalar multiple of b. The modes under consideration are therefore GL and the result follows from GL 1.

Lastly, we have a counterpart of GT 3.

GL 5. *If, for some $\boldsymbol{n}$ and $\chi > 0$, a mode of thermoelastic wave propagation has a real displacement amplitude, this mode is either GL or GT.*

In equation (3.9) we replace U by U_k and set $\mathrm{Re}\, w_k = w_k^+$, $\mathrm{Im}\, w_k = w_k^-$. When U_k

is real the imaginary part of this equation is

$$w_k^- U_k = -w_k^+ \chi \{w_k^{+2} + (w_k^- + \chi)^2\}^{-1} (b^{\mathrm{T}} U_k) b.$$

If $w_k^- \neq 0$, U_k is a scalar multiple of b and the mode is GL. If $w_k^- = 0$, $b^{\mathrm{T}} U_k = 0$ and the mode is GT.

If, for some $\boldsymbol{n}$ and $\chi > 0$, the displacement amplitude U_k of a mode is a scalar multiple of its limiting value as $\chi \to 0$ or as $\chi \to \infty$, no generality is lost in taking U_k to be real. By GL 5, the mode is thus either GL or GT. If the limiting values are equal, $\hat{Q}$ and $\tilde{Q}$ have a common eigenvector which must be orthogonal to b or a scalar multiple of b (Chadwick & Currie 1974*b*, Appendix). In the former case, (3.1) holds and a GT mode exists. In the latter, GL 2 requires that $\boldsymbol{n}$ define a GL normal. We thus arrive at conditions analogous to (3.4) and (3.5) sufficient for the existence of either a GL or a GT mode.

(*c*) *Repeated eigenvalues of $\tilde{Q}$ and $\hat{Q}$*

We conclude this section by considering the consequences of one or both of the dimensionless acoustical tensors $\tilde{Q}$ and $\hat{Q}$ having an eigenvalue of multiplicity 2.

If, for some $\boldsymbol{n}$, the eigenvalues of $\tilde{Q}$ satisfy $\tilde{w}_1 = \tilde{w}_2 = \tilde{w}_d \neq \tilde{w}_3$ and $\tilde{q}_3$ is a unit eigenvector associated with $\tilde{w}_3$, $\tilde{Q}$ has the spectral representation

$$\tilde{Q} = \tilde{w}_d I - (\tilde{w}_d - \tilde{w}_3) \tilde{q}_3 \tilde{q}_3^{\mathrm{T}}. \tag{3.10}$$

Likewise, with similar notation, if $\hat{w}_1 = \hat{w}_2 = \hat{w}_d \neq \hat{w}_3$,

$$\hat{Q} = \hat{w}_d I - (\hat{w}_d - \hat{w}_3) \hat{q}_3 \hat{q}_3^{\mathrm{T}}. \tag{3.11}$$

When (3.10) applies and b is not parallel to $\tilde{q}_3$, $\tilde{Q}b$ and $\tilde{Q}^2 b$ are linear combinations of b and $\tilde{q}_3$. The condition (3.1) is therefore fulfilled and one of the modes with wave normal $\boldsymbol{n}$ is GT. It can easily be verified from (2.24), $(2.25)_1$ and (3.10) that the modal root and displacement amplitude of the GT mode are $w_k = \tilde{w}_d$, $U_k = \tilde{q}_3 \times b$. When b is orthogonal to $\tilde{q}_3$, b is an eigenvector of $\tilde{Q}$ and, by GL 2, $\boldsymbol{n}$ defines a GL normal. For the second GT mode, $w_k = \tilde{w}_3$, $U_k = \tilde{q}_3$, and the modal roots of the GL modes are given by equation (3.8) with $\tilde{w}_d$ in place of $\tilde{w}_1$. GL 2 also ensures that $\boldsymbol{n}$ is directed along a GL normal when b is parallel to $\tilde{q}_3$. $\tilde{w}_1$ is now replaced by $\tilde{w}_3$ in (3.8) and, for both GT modes, the modal root is $\tilde{w}_d$. We return to this situation in § 5 *c*.

The foregoing conclusions remain valid when (3.11) holds, with obvious changes of detail. In view of the relation (2.15), $\hat{w}_d - b^{\mathrm{T}} b$ or $\hat{w}_3 - b^{\mathrm{T}} b$ must be substituted for $\tilde{w}_1$ in (3.8) according as b is orthogonal or parallel to $\hat{q}_3$.

Turning to the case in which (3.10) and (3.11) both apply, we deduce from (2.15) that

$$(\hat{w}_d - \tilde{w}_d) I - (\hat{w}_d - \hat{w}_3) \hat{q}_3 \hat{q}_3^{\mathrm{T}} + (\tilde{w}_d - \tilde{w}_3) \tilde{q}_3 \tilde{q}_3^{\mathrm{T}} - b b^{\mathrm{T}} = 0. \tag{3.12}$$

Suppose first that $\hat{w}_d \neq \tilde{w}_d$. Letting the left-hand side of (3.12) act in turn on $\hat{q}_3 \times \tilde{q}_3$, $\hat{q}_3 \times b$, $\tilde{q}_3 \times b$, we obtain

$$\left.\begin{aligned} (\hat{w}_d - \tilde{w}_d) \hat{q}_3 \times \tilde{q}_3 &= [b, \hat{q}_3, \tilde{q}_3] b, \\ (\hat{w}_d - \tilde{w}_d) \hat{q}_3 \times b &= (\tilde{w}_d - \tilde{w}_3) [b, \hat{q}_3, \tilde{q}_3] \tilde{q}_3, \\ (\hat{w}_d - \tilde{w}_d) \tilde{q}_3 \times b &= (\hat{w}_d - \hat{w}_3) [b, \hat{q}_3, \tilde{q}_3] \hat{q}_3. \end{aligned}\right\} \tag{3.13}$$

If $[b, \hat{q}_3, \tilde{q}_3] = 0$, the vector products in (3.13) vanish and b, $\hat{q}_3$, $\tilde{q}_3$ are scalar multiples of one another. But then (3.12) gives $(\hat{w}_d - \tilde{w}_d)r = 0$, where r is a non-zero vector orthogonal to each of b, $\hat{q}_3$, $\tilde{q}_3$, and this is false. Hence $[b, \hat{q}_3, \tilde{q}_3] \neq 0$ and equations (3.13) tell us that b, $\hat{q}_3$, $\tilde{q}_3$ are mutually orthogonal and $\tilde{w}_3 = \hat{w}_d$, $\hat{w}_3 = \tilde{w}_d$, $\hat{w}_d = \tilde{w}_d + b^{\mathrm{T}}b$. As shown in the third paragraph of this subsection, $\boldsymbol{n}$ defines a GL normal. The parameters of the GT modes are $\tilde{w}_d$, $\hat{q}_3$, and $\hat{w}_d$, $\tilde{q}_3$ and the modal roots of the GL modes are specified by equation (3.8) with $\tilde{w}_d$ replacing $\tilde{w}_1$.

When $\hat{w}_d = \tilde{w}_d = w_d$ the action of the left-hand side of (3.12) on b, $\hat{q}_3$, $\tilde{q}_3$ in turn provides the linear dependences,

$$(w_d - \hat{w}_3)(b^{\mathrm{T}}\hat{q}_3)\hat{q}_3 - (w_d - \tilde{w}_3)(b^{\mathrm{T}}\tilde{q}_3)\tilde{q}_3 + (b^{\mathrm{T}}b)b = 0, \tag{3.14}$$

$$(w_d - \hat{w}_3)\hat{q}_3 - (w_d - \tilde{w}_3)(\hat{q}_3^{\mathrm{T}}\tilde{q}_3)\tilde{q}_3 + (b^{\mathrm{T}}\hat{q}_3)b = 0, \tag{3.15}$$

$$(w_d - \hat{w}_3)(\hat{q}_3^{\mathrm{T}}\tilde{q}_3)\hat{q}_3 - (w_d - \tilde{w}_3)\tilde{q}_3 + (b^{\mathrm{T}}\tilde{q}_3)b = 0. \tag{3.16}$$

The consistency of equations (3.14), (3.15) and (3.15), (3.16) requires that $(b^{\mathrm{T}}\hat{q}_3)^2 = b^{\mathrm{T}}b$, i.e. $(\hat{q}_3 \times b)^{\mathrm{T}}(\hat{q}_3 \times b) = 0$, and $(\hat{q}_3^{\mathrm{T}}\tilde{q}_3)^2 = 1$. Hence $b = \pm(b^{\mathrm{T}}b)^{1/2}\hat{q}_3$, $\tilde{q}_3 = \pm\hat{q}_3$ and we conclude that b, $\hat{q}_3$, $\tilde{q}_3$ are scalar multiples of one another. From (3.12), $\hat{w}_3 = \tilde{w}_3 + b^{\mathrm{T}}b$. We thus meet again the state of affairs in which $\boldsymbol{n}$ determines a GL normal and the GT modes have the same modal root.

4. Degeneracy of modal roots

(*a*) *Degeneracy conditions*

If, for some values of $\boldsymbol{n}$ and χ, the secular equation (2.14) has a doubly degenerate root,

$$w_j = w_k = w_d, \tag{4.1}$$

there holds, in addition to

$$\det S(w_d) = 0, \tag{4.2}$$

the condition

$$\frac{\mathrm{d}}{\mathrm{d}w}\det S(w)|_{w=w_d} = 0. \tag{4.3}$$

The formula

$$\frac{\mathrm{d}}{\mathrm{d}x}\det M(x) = \mathrm{tr}\left\{M^*(x)\frac{\mathrm{d}}{\mathrm{d}x}M(x)\right\}, \tag{4.4}$$

valid for an arbitrary differentiable symmetric matrix function (see, for example, Chadwick 1976, p. 20), enables us to rewrite equation (4.3) as

$$\mathrm{tr}\left\{S^*(w_d)\frac{\mathrm{d}}{\mathrm{d}w}S(w)|_{w=w_d}\right\} = 0. \tag{4.5}$$

Equations $(\mathrm{A}\,6)_{1,2}$, (A 9) and $(\mathrm{A}\,10)_4$, in conjunction with (4.2), provide the alternative forms

$$\mathrm{tr}\,S^*(w_d) - b^{\mathrm{T}}(w_dI - \tilde{Q})^*b = 0 \tag{4.6}$$

and

$$\mathrm{tr}\,S^*(w_d) - w_d^{-1}(w_d + \mathrm{i}\chi)\det(w_dI - \tilde{Q}) = 0 \tag{4.7}$$

of (4.5), and $(\mathrm{A}\,5)_2$, (A 9) and $(\mathrm{A}\,10)_2$ yield the identity,

$$\mathrm{tr}\,S^*(w) = \mathrm{i}\chi\,\mathrm{tr}(wI - \tilde{Q})^* + w\,\mathrm{tr}(wI - \hat{Q})^* + \det(wI - \tilde{Q}). \tag{4.8}$$

Equations (4.5), (4.6) and (4.8) were first given by Scott (1993). We note, however, that the plus sign on the right-hand side of Scott's equation (4.15) should be minus (with a corresponding correction in (4.14)) and that the final $\hat{Q}$ in his equation (4.16) (and the preceding unnumbered expression for S^*_{44}) should be $\tilde{Q}$.

By virtue of equation (2.27), $S(w)$ can be replaced by $T(w)$ in the condition (4.5), and a direct algebraic justification of this step is furnished by equations (A 8), $(\mathrm{A}\,10)_{3,4}$ and (4.2). Equations $(\mathrm{A}\,6)_3$, (A 9), $(\mathrm{A}\,10)_4$ and (4.2) supply the further version

$$\operatorname{tr} T^*(w_d) - \mathrm{i}\chi w_d^{-1}(w_d + \mathrm{i}\chi)^{-1} \det(w_d I - \tilde{Q}) = 0 \tag{4.9}$$

of (4.5) and, from (A 7) and $(\mathrm{A}\,10)_2$,

$$\operatorname{tr} T^*(w) = (w + \mathrm{i}\chi)^{-1}\{\mathrm{i}\chi \operatorname{tr}(wI - \tilde{Q})^* + w \operatorname{tr}(wI - \hat{Q})^*\}. \tag{4.10}$$

(b) *Possible forms of degenerate modes*

On account of (4.2), the rank of the 4×4 matrix $S(w_d)$ is at most 3. From (2.26), $w_d \neq -\mathrm{i}\chi$ and it follows from the definition (2.12) that the rank is at least 1. If the rank is 1, we infer from (2.12) that w_d is an eigenvalue of $\tilde{Q}$, and hence real, and all the elements of $S(w_d)$ except the (4, 4) entry $w_d + \mathrm{i}\chi$ are therefore real. Since $w_d^{1/2} b \neq 0$, it is clearly impossible, in this situation, for all the rows of $S(w_d)$ to be scalar multiples of one another. The rank is thus 2 or 3 and there stem from these possibilities different forms of dependence on x of the two modes involved in the degeneracy (4.1).

When

$$\operatorname{rank} S(w_d) = 2, \tag{4.11}$$

the null space of $S(w_d)$ is two dimensional and equation (2.20), with $w_k = w_d$, has two linearly independent solutions. The degenerate modes therefore retain the form (2.19) with constant displacement and scaled temperature change amplitudes.

When

$$\operatorname{rank} S(w_d) = 3, \tag{4.12}$$

the null space of $S(w_d)$ is one dimensional and, as allowed by the general solution (2.10), (2.18), there is a composite degenerate mode of the form

$$v_d = \exp\{\mathrm{i}\omega(s_d x - t)\}(V^{(1)} + \mathrm{i}\omega s_d x V^{(2)}), \tag{4.13}$$

with

$$s_d = (\gamma w_d / \bar{\rho})^{-1/2}. \tag{4.14}$$

On substituting the expression (4.13) into equation (2.8) and utilizing (2.9) and (2.13), we find that

$$S(w_d)V^{(1)} = WV^{(2)}, \qquad S(w_d)V^{(2)} = 0, \tag{4.15}$$

where

$$W = \begin{bmatrix} w_d\{2I - (w_d + \mathrm{i}\chi)^{-1} b b^{\mathrm{T}}\} & 0 \\ 0 & w_d - \mathrm{i}\chi \end{bmatrix} \tag{4.16}$$

and equations $(4.15)_2$ and (2.24) (in the form $T(w_d)U^{(2)} = 0$) have been used to simplify W. Because of the symmetry of $S(w_d)$, equations (4.15) together imply that

$$V^{(2)\mathrm{T}} W V^{(2)} = 0. \tag{4.17}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

According to $(4.15)_2$ and (4.17), $V^{(2)}$ is orthogonal to each row of $S(w_d)$ and to $WV^{(2)}$. The augmented 5×4 matrix formed by adjoining the row vector $V^{(2)\mathrm{T}}W$ to $S(w_d)$ thus has the same rank as $S(w_d)$ and the consistency of the system of four algebraic equations represented by $(4.15)_1$ is assured.

It may be mentioned in passing that the degenerate mode (4.13) can also be reached by a limiting procedure. If, for some value of χ, the wave normals of two modes with modal roots w_j, w_k enclose an angle φ and $w_j \to w_d$, $w_k \to w_d$ as $\varphi \to 0$, equation (2.20) can be expanded in powers of φ for each mode. Differencing the expansions, dividing by φ and passing to the limit $\varphi \to 0$ reproduces equation $(4.15)_1$, with $V^{(2)}$ proportional to the derivative $\mathrm{d}(V_j - V_k)/\mathrm{d}\varphi$ evaluated at $\varphi = 0$. It is assumed that $\mathrm{d}(V_j - V_k)/\mathrm{d}\varphi \neq 0$ at $\varphi = 0$: otherwise second derivatives with respect to φ are involved.

In parallel with (2.22), the general solution of equation $(4.15)_2$ is

$$V^{(2)} = [U^{(2)\mathrm{T}}, \varPhi^{(2)}]^{\mathrm{T}}$$

with

$$U^{(2)} = -a_1 w_d^{1/2}(w_d I - \tilde{Q})^* b, \qquad \varPhi^{(2)} = a_1 \det(w_d I - \tilde{Q}), \tag{4.18}$$

a_1 being an arbitrary constant. Substituting from (4.16) and (4.18) into equation (4.17) and using (2.16), with $w = w_d$, we obtain the compatibility condition

$$(w_d + \mathrm{i}\chi)^2 b^{\mathrm{T}}(w_d I - \tilde{Q})^{*2} b = \mathrm{i}\chi\{b^{\mathrm{T}}(w_d I - \tilde{Q})^* b\}^2. \tag{4.19}$$

By analogy with (2.24) and (2.23), equations (4.18) and (4.19) yield the relations

$$T(w_d)U^{(2)} = 0, \qquad \varPhi^{(2)} = -w_d^{1/2}(w_d + \mathrm{i}\chi)^{-1} b^{\mathrm{T}} U^{(2)}, \tag{4.20}$$

and, by appeal to (4.19) and (2.16), we deduce from (4.18) and $(4.20)_2$ that

$$U^{(2)\mathrm{T}}U^{(2)} = \mathrm{i}\chi(w_d + \mathrm{i}\chi)^{-2}(b^{\mathrm{T}}U^{(2)})^2 = \mathrm{i}\chi w_d^{-1}\varPhi^{(2)2}. \tag{4.21}$$

It can be verified, with the aid of equations (2.12), (4.16), (4.18), (2.16) and (4.19), that a particular solution of $(4.15)_1$ is $V^{(1)} = [U^{(1)\mathrm{T}}, \varPhi^{(1)}]^{\mathrm{T}}$ with

$$U^{(1)} = a_1 w_d^{1/2}[(w_d I - \tilde{Q})^* - 2(w_d + \mathrm{i}\chi)\{b^{\mathrm{T}}(w_d I - \tilde{Q})^* b\}^{-1}(w_d I - \tilde{Q})^{*2}]b, \quad \varPhi^{(1)} = 0. \tag{4.22}$$

The general solution is secured by adding

$$a_2[-w_d^{1/2}\{(w_d I - \tilde{Q})^* b\}^{\mathrm{T}}, \det(w_d I - \tilde{Q})]^{\mathrm{T}} \tag{4.23}$$

to $V^{(1)}$, a_2 being a second arbitrary constant. Corresponding to equations (4.20), we find that

$$\left.\begin{aligned} T(w_d)U^{(1)} &= 2w_d\{I - \mathrm{i}\chi(w_d + \mathrm{i}\chi)^{-2} bb^{\mathrm{T}}\}U^{(2)}, \\ \varPhi^{(1)} &= -w_d^{1/2}(w_d + \mathrm{i}\chi)^{-2}\{(w_d + \mathrm{i}\chi)b^{\mathrm{T}}U^{(1)} + (w_d - \mathrm{i}\chi)b^{\mathrm{T}}U^{(2)}\}, \end{aligned}\right\} \tag{4.24}$$

and (4.20) and (4.24) can be derived directly from (4.15). It follows from equations $(4.20)_1$ and $(4.24)_1$ that $U^{(1)}$ can be a scalar multiple of $U^{(2)}$ only if both are scalar multiples of b, that is only if $\boldsymbol{n}$ defines a GL normal. If two GT modes are degenerate, the condition (4.11) necessarily holds and the displacement amplitudes remain constant. An example of a degeneracy of GL modes is discussed in § 6 *c*.

The disposable constants a_1 and a_2 appearing in the amplitude $V^{(1)} + \mathrm{i}\omega s_d x V^{(2)}$ of the degenerate mode (4.13) may be specified by boundary conditions. In circumstances forcing a_1 to be zero, (4.13) reduces to the constant amplitude form (2.19).

5. Degeneracy of the zero eigenvalue of $S(w_k)$ and $T(w_k)$

(a) Degeneracy conditions for $S(w_k)$

To each modal root w_k there corresponds a matrix $S(w_k)$ possessing a zero eigenvalue. For some values of $\boldsymbol{n}$ and χ this eigenvalue may be degenerate. Then, in addition to (2.21), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\det\{sI - S(w_k)\}|_{s=0} = 0.$$

The application of the formula (4.4) brings this condition into the form,

$$\operatorname{tr} S^*(w_k) = 0. \tag{5.1}$$

Equation (5.1) was obtained by Scott (1993, § 2) as a necessary and sufficient condition for $S(w_k)$ to have an isotropic null vector, that is a vector V_k satisfying (2.20) and such that $V_k^{\mathrm{T}}V_k = 0$. The approach followed here of associating (5.1) with a repeated zero eigenvalue of $S(w_k)$ allows us to distinguish between the two types of eigenvalue degeneracy which a non-Hermitian complex matrix can exhibit. These are *non-semisimple degeneracy*, for which

$$\operatorname{tr} S^*(w_k) = 0, \qquad S^*(w_k) \neq 0. \tag{5.2}$$

and *semisimple degeneracy*, for which

$$S^*(w_k) = 0. \tag{5.3}$$

We consider these possibilities in the next two subsections.

(b) Non-semisimple degeneracy of $S(w_k)$

In this, the more general, type of degeneracy, $S(w_k)$ does not possess a complete set of eigenvectors, and consequently there is no basis in which $S(w_k)$ can be diagonalized. The degenerate zero eigenvalue, taken to be of multiplicity 2, is associated with an isotropic eigenvector V_k and a generalized eigenvector V_g satisfying $S(w_k)V_g = V_k$ (see, for example, Pease 1965, ch. III). The canonical form of $S(w_k)$ in the basis formed by three linearly independent eigenvectors and V_g is diagonal, apart from a 2×2 Jordan box in the eigenspace spanned by V_k and V_g. This box has one non-zero element which occupies an off-diagonal place. Thus $S(w_k)$ has a non-zero 3×3 minor and $(5.2)_2$ follows.

(c) Semisimple degeneracy of $S(w_k)$

In this case $S(w_k)$ has four linearly independent eigenvectors, constituting a basis for its diagonalization, and since two of the diagonal entries in the canonical form are zero, equation (5.3) necessarily holds. Among the base vectors are two null vectors corresponding to the zero eigenvalue and chosen as any pair of linearly independent vectors from the two-dimensional eigenspace.

With reference to equations (A 4), (A 7), (A 9), and $(\mathrm{A}\,10)_2$, we can express the matrix condition (5.3) as

$$\left.\begin{aligned} \mathrm{i}\chi(w_kI - \tilde{Q})^* + w_k(w_kI - \hat{Q})^* = 0,\\ (w_kI - \tilde{Q})^*b = 0, \quad \det(w_kI - \tilde{Q}) = 0. \end{aligned}\right\} \tag{5.4}$$

Equation $(5.4)_3$ states that w_k is an eigenvalue of $\tilde{Q}$ and is therefore real, positive and independent of χ. Equation $(5.4)_1$ thereupon splits into

$$(w_k I - \tilde{Q})^* = 0, \qquad (w_k I - \hat{Q})^* = 0, \tag{5.5}$$

equation $(5.5)_1$ subsuming $(5.4)_2$. We conclude that w_k is a repeated eigenvalue of both $\tilde{Q}$ and $\hat{Q}$, whence, as shown at the end of § 3 *c*, $\boldsymbol{n}$ defines a GL normal for which the GT modes have the same modal root.

A GL normal with this property is from now on called a G-*modal axis*, by analogy with the acoustic axis of classical elastodynamics, characterized in § 1. It should be emphasized that the G-modal axis and the GL normal are non-dispersive features insofar as the wave normal which defines them is independent of χ.

We have proved above

GM 1. *A wave normal $\boldsymbol{n}$ for which there is a modal root w_k such that the zero eigenvalue of $S(w_k)$ is doubly semisimply degenerate is directed along a G-modal axis.*

Also, rephrasing conclusions reached in § 3 *c*, we have

GM 2. *If, for some $\boldsymbol{n}$, either $\tilde{Q}$ or $\hat{Q}$ has a repeated eigenvalue and b is an eigenvector associated with the non-degenerate eigenvalue, then $\boldsymbol{n}$ defines a G-modal axis.*

GM 3. *If, for some $\boldsymbol{n}$, $\tilde{Q}$ and $\hat{Q}$ have a common repeated eigenvalue (or, equivalently, a GT mode has a degenerate modal root), then $\boldsymbol{n}$ defines a G-modal axis.*

Equations (5.5) imply (5.4) and are therefore necessary as well as sufficient conditions for $\boldsymbol{n}$ to specify a G-modal axis. They are satisfied when there is no pre-stress and the heat-conducting elastic material has a 3- or 4-fold rotation axis or an axis of transverse isotropy with which $\boldsymbol{n}$ is aligned. Otherwise, as suggested by the existence theorem for a GL normal quoted in § 3 *b*, the conditions (5.5) for a G-modal axis may be fulfilled by some adventitious combination of $\boldsymbol{n}$, the pre-stress and the material constants.

(*d*) *Degeneracy of the zero eigenvalue of* $T(w_k)$

When w_k is a modal root,

$$\det T(w_k) = 0 \tag{5.6}$$

and the matrix $T(w_k)$ has a zero eigenvalue. As in the case of $S(w_k)$, a degeneracy of this eigenvalue may be non-semisimple or semisimple. For the former type,

$$\operatorname{tr} T^*(w_k) = 0, \qquad T^*(w_k) \neq 0, \tag{5.7}$$

and for the latter,

$$T^*(w_k) = 0. \tag{5.8}$$

In connexion with these conditions we take note of the relations,

$$\det T(w) = \det(wI - \tilde{Q}) - w(w + \mathrm{i}\chi)^{-1} b^{\mathrm{T}} T^*(w) b, \tag{5.9}$$

$$T^*(w) = (w + \mathrm{i}\chi)^{-1}\{\mathrm{i}\chi(wI - \tilde{Q})^* + w(wI - \hat{Q})^*\}, \tag{5.10}$$

$$\operatorname{tr} S^*(w) = (w + \mathrm{i}\chi) \operatorname{tr} T^*(w) + \det(wI - \tilde{Q})^*, \tag{5.11}$$

obtained from equations (A 2), (A 7), $(\mathrm{A}\,5)_1$, (A 9) and $(\mathrm{A}\,10)_2$.

If the zero eigenvalue of $T(w_k)$ is semisimply degenerate, we deduce from equations (5.9), (5.6) and (5.8) that $\det(wI - \tilde{Q}) = 0$ and hence that w_k is real. Equations (5.10) and (5.8) then yield (5.5) which entail (5.3) and the conclusion that the zero eigenvalue of $S(w_k)$ is also semisimply degenerate. The converse is also true since (5.1) implies (5.5) and then, through (5.10), (5.8).

In contrast, non-semisimple degeneracies of the zero eigenvalues of $S(w_k)$ and $T(w_k)$ cannot coexist. Were this not so, (5.2) and (5.7) would hold simultaneously and equations (5.11), (2.21) and (2.17) would require that

$$\det(w_k I - \tilde{Q}) = 0, \qquad \det(w_k I - \hat{Q}) = 0. \tag{5.12}$$

Thus w_k would be real and (4.10) and $(5.7)_1$ would lead to

$$\operatorname{tr}(w_k I - \tilde{Q})^* = 0, \qquad \operatorname{tr}(w_k I - \hat{Q})^* = 0. \tag{5.13}$$

For a real symmetric 3×3 matrix the vanishing of the determinant and the trace of the adjugate ensure the vanishing of the adjugate. By (5.4), $S^*(w_k)$ would therefore be zero, contravening the condition $(5.2)_2$.

6. Relationship of modal root and eigenvalue degeneracies

It has been shown in § 4 *a* that a degeneracy of a modal root has two possible forms, depending on the rank of $S(w_d)$ being 2 or 3. In § 5 *a* a degeneracy of the zero eigenvalue of $S(w_k)$ has also been found to offer two alternatives: $S(w_k)$ is either non-semisimple or semisimple. Including cases in which one type of degeneracy is non-existent, there are thus eight combinations of modal root and eigenvalue degeneracies. We prove in this section that only three of them actually occur.

(*a*) *Coexistent degeneracies*

Suppose that, for some values of $\boldsymbol{n}$ and χ, there is a modal root degeneracy, given by (4.1), for which equation (4.11) also holds, implying that the degenerate modes have constant amplitudes. Then the zero eigenvalue of $S(w_d)$ is degenerate, since otherwise the rank of $S(w_d)$ would be 3. By (4.11), $S^*(w_d) = 0$. The coexistent eigenvalue degeneracy is therefore semisimple. As shown in § 5 *c*, $\boldsymbol{n}$ defines a G-modal axis and the degenerate modes are GT.

Starting from the assumption that, for some $\boldsymbol{n}$, χ, a semisimple degeneracy of the zero eigenvalue of $S(w_k)$ exists, w_k being a modal root, we deduce from GM 1 in § 5 *c* that $\boldsymbol{n}$ specifies a G-modal axis. There is hence a degeneracy of two GT modes and $w_k = w_d$, the common modal root. From (5.3), $S^*(w_d) = 0$, so equation (4.11) applies and the degenerate modes have constant amplitudes.

A modal root degeneracy for which (4.11) holds can therefore exist if and only if the zero eigenvalue of $S(w_d)$ is semisimply degenerate, and when these coexistent degeneracies occur the wave normal is directed along a G-modal axis.

If, for some $\boldsymbol{n}$, there exists a degenerate modal root w_d which is independent of the dimensionless frequency χ, equations (2.17), (4.7) and (4.8) give

$$\mathrm{i}\chi \det(w_d I - \tilde{Q}) + w_d \det(w_d I - \hat{Q}) = 0,$$

$$\mathrm{i}\chi \operatorname{tr}(w_d I - \tilde{Q})^* + w_d \operatorname{tr}(w_d I - \hat{Q})^* - \mathrm{i}\chi w_d^{-1} \det(w_d I - \tilde{Q}) = 0.$$

These equations hold identically in χ, so (5.12) and (5.13) follow, with w_d in place of w_k. Thus $S^*(w_d) = 0$. We conclude that $\boldsymbol{n}$ defines a G-modal axis which, by this argument, is the *only* degeneracy of plane harmonic thermoelastic waves which is non-dispersive in the sense specified in § 5 *c*.

(*b*) *Single degeneracies*

There remains only one pair of degeneracies which could coexist, a degeneracy of a modal root for which (4.1) and (4.12) hold and a non-semisimple degeneracy of the zero eigenvalue of $S(w_d)$. For the first of these, equations (4.7) and (4.9) yield

$$\mathrm{i}\chi \operatorname{tr} S^*(w_d) = (w_d + \mathrm{i}\chi)^2 \operatorname{tr} T^*(w_d).$$

For the eigenvalue degeneracy, equation $(5.2)_1$ gives $\operatorname{tr} S^*(w_d) = 0$, whence

$$\operatorname{tr} S^*(w_d) = 0, \qquad \operatorname{tr} T^*(w_d) = 0. \tag{6.1}$$

As shown at the end of § 5 *d*, however, equations (6.1) imply that $S^*(w_d) = 0$ and this is incompatible with (4.12). The degeneracies in question can therefore exist only on their own, and these are the second and third of the possibilities mentioned in the opening paragraph of the section.

(*c*) *An example: degenerate GL modes*

It has been shown in the previous subsections that a degeneracy of a modal root, w_d, can arise in two ways. First, when the wave normal defines a G-modal axis, the zero eigenvalues of $S(w_d)$ and $T(w_d)$ are semisimply degenerate, and the degenerate modes are GT with constant displacement amplitudes orthogonal to b. Second, when the zero eigenvalues of $S(w_d)$ and $T(w_d)$ are non-degenerate and the amplitude of the composite degenerate mode is a linear function of distance x along the wave normal. The second alternative is the one which normally occurs and we exemplify it by returning to the discussion of GL modes in § 3 *b*.

The relevant modal roots are degenerate when equation (3.8) has equal roots. The discriminant of the quadratic vanishes if and only if

$$\tilde{w}_1 = \varepsilon, \qquad \chi = 2\varepsilon, \tag{6.2}$$

where

$$\varepsilon = b^{\mathrm{T}} b, \tag{6.3}$$

and the repeated root is

$$w_d = (1 - \mathrm{i})\varepsilon. \tag{6.4}$$

By virtue of equation $(2.6)_3$, the definition (6.3) can be rewritten as

$$\varepsilon = (\bar{\rho} c \gamma / \bar{T})^{-1} |\boldsymbol{\beta n}|^2, \tag{6.5}$$

and in this guise ε is recognized as the thermoelastic coupling constant (Chadwick 1979, p. 211). Since ε is typically in the interval $(10^{-4}, 10^{-1})$ (Chadwick 1960, table 1), it is evident from equations (6.2) that we have chosen an example of mathematical rather than physical interest.

In relation to the orthonormal basis $\{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3\}$ of unit eigenvectors of $\tilde{Q}$, we deduce from equations $(6.2)_1$ and (6.4) that

$$w_d I - \tilde{Q} = \operatorname{diag}(-\mathrm{i}\varepsilon, \tau_2, \tau_3), \tag{6.6}$$

where

$$\tau_i = (1 - \mathrm{i})\varepsilon - \tilde{w}_i, \qquad i = 2, 3.$$

Substituting from (6.6), (6.4) and $(6.2)_2$ into equation (2.12) and recalling that

$b = (b^{\mathrm{T}}b)^{1/2}\tilde{q}_1 = \varepsilon^{1/2}\tilde{q}_1$, we obtain

$$S(w_d) = \begin{bmatrix} -\mathrm{i}\varepsilon & 0 & 0 & (1-\mathrm{i})^{1/2}\varepsilon \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ (1-\mathrm{i})^{1/2}\varepsilon & 0 & 0 & (1+\mathrm{i})\varepsilon \end{bmatrix}.$$

It follows that

$$\det\{sI_4 - S(w_d)\} = s(s-\varepsilon)(s-\tau_2)(s-\tau_3),$$

from which we confirm that $\operatorname{rank} S(w_d) = 3$ and the zero eigenvalue of $S(w_d)$ is non-degenerate.

The displacement and scaled temperature change amplitudes of the composite degenerate GL mode can now be determined from the general solution given in § 4 *b*. From (6.6),

$$\left.\begin{aligned} (w_dI - \tilde{Q})^*b = \varepsilon^{1/2}\tau_2\tau_3\tilde{q}_1, \qquad \det(w_dI - \tilde{Q}) = -\mathrm{i}\varepsilon\tau_2\tau_3, \\ b^{\mathrm{T}}(w_dI - \tilde{Q})^*b = \varepsilon\tau_2\tau_3, \qquad (w_dI - \tilde{Q})^{*2}b = \varepsilon^{1/2}\tau_2^2\tau_3^2\tilde{q}_1, \end{aligned}\right\} \tag{6.7}$$

and it is clear from (6.7), $(6.2)_2$ and (6.4) that the compatibility condition (4.19) is satisfied. The same equations, entered into (4.18) and (4.22), give

$$U^{(2)} = c_1(1-\mathrm{i})^{1/2}\tilde{q}_1, \quad \varPhi^{(2)} = c_1\mathrm{i}; \qquad U^{(1)} = c_1(1+2\mathrm{i})(1-\mathrm{i})^{1/2}\tilde{q}_1, \quad \varPhi^{(1)} = 0, \tag{6.8}$$

with

$$c_1 = -a_1\varepsilon\tau_2\tau_3,$$

and it may readily be verified, with the aid of equations $(6.2)_2$ and (6.4), that the expressions for $U^{(2)}$ and $\varPhi^{(2)}$ meet the requirements (4.21). With reference to equations (4.13) and (4.23), we can finally assemble the solution from (6.8) as

$$\left.\begin{aligned} u_d = \exp\{\mathrm{i}\omega(s_dx - t)\}(1-\mathrm{i})^{1/2}\{c_1(1+2\mathrm{i}+\mathrm{i}\omega s_dx) + c_2\}\tilde{q}_1, \\ \phi_d = \exp\{\mathrm{i}\omega(s_dx - t)\}(-c_1\omega s_dx + \mathrm{i}c_2). \end{aligned}\right\} \tag{6.9}$$

The slowness s_d is given by equations (4.14) and (6.4) as

$$s_d = \tfrac{1}{2}(\varepsilon\gamma/\bar{\rho})^{-1/2}\{(2^{1/2}-1)^{-1/2} + (2^{1/2}-1)^{1/2}\mathrm{i}\}. \tag{6.10}$$

This example has been discussed previously by Chadwick (1973, § 4(c)) as a critical case in a general study of GL modes. Chadwick's results include (6.10), but not (6.9).

7. Small perturbations of the isentropic limit

Up to this point our discussion of the modes of thermoelastic wave propagation has been exact. We now embark on an approximate study of degeneracies occurring in a small neighbourhood of the isentropic limit $\chi \to 0$ and a wave normal $\boldsymbol{n}_0$ for which the isentropic acoustical tensor has a repeated eigenvalue. The physically realistic nature of the low-frequency regime $\chi \ll 1$ has been pointed out in § 2 *d*. Our approach is based on perturbation expansions in combinations of powers of χ and a quantity related to the magnitude of

$$\Delta\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{n}_0. \tag{7.1}$$

(*a*) *The unperturbed state*

As explained in § 2 *d*, isentropic conditions prevail in the low-frequency limit $\chi \to 0$, and with each unit vector $\boldsymbol{n}$ there are associated three plane harmonic waves having $\boldsymbol{n}$ as wave normal. Their speeds of propagation are $(\gamma \hat{w}_i/\bar{\rho})^{1/2}$, $i = 1, 2, 3$, where $\hat{w}_i$ are the eigenvalues of the dimensionless isentropic acoustical tensor $\hat{\boldsymbol{Q}}$, and their displacement amplitudes are scalar multiples of corresponding unit eigenvectors $\hat{\boldsymbol{q}}_i$. Since $\hat{\boldsymbol{Q}}$ is positive definite (see § 2 *b*), $\hat{w}_i > 0$.

We assume that there exists, in the isentropic limit, a unit vector $\boldsymbol{n}_0$ defining an acoustic axis. This means that two plane harmonic waves with wave normal $\boldsymbol{n}_0$ have equal speeds of propagation and hence that $\hat{\boldsymbol{Q}}_0$, the value of $\hat{\boldsymbol{Q}}$ at $\boldsymbol{n} = \boldsymbol{n}_0$, has an eigenvalue $\hat{w}_0$ of multiplicity 2. In the approximate analysis developed in this and the next section the unperturbed state is specified by $\chi = 0$, $\Delta \boldsymbol{n} = \boldsymbol{0}$.

The typical stress γ_0 involved in the definition $(2.6)_2$ of $\hat{\boldsymbol{Q}}$ is evaluated at $\boldsymbol{n} = \boldsymbol{n}_0$ and, for simplicity, we use the constant γ_0 rather than the $\boldsymbol{n}$-dependent γ for purposes of non-dimensionalization. In particular, the counterpart of equation $(2.6)_3$ is, in direct notation,

$$\boldsymbol{b}_0 = (\bar{\rho} c \gamma_0 / \bar{T})^{-1/2} \boldsymbol{\beta} \boldsymbol{n}_0.$$

By GM 2 in § 5 *c*, the degeneracy in the unperturbed state extends to all frequencies at $\boldsymbol{n} = \boldsymbol{n}_0$ if $\boldsymbol{b}_0$ is a scalar multiple of $\hat{\boldsymbol{q}}_{03}$, a unit eigenvector of $\hat{\boldsymbol{Q}}_0$ associated with the non-degenerate eigenvalue $\hat{w}_{03}$, $\boldsymbol{n}_0$ then defining a G-modal axis. We avoid this simple case by assuming henceforth that $\boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03} \neq \boldsymbol{0}$. It is then convenient to employ the orthonormal basis $q = \{\hat{\boldsymbol{q}}_{01}, \hat{\boldsymbol{q}}_{02}, \hat{\boldsymbol{q}}_{03}\}$ in which

$$\hat{\boldsymbol{q}}_{01} = |\boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03}|^{-1} \boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03}, \qquad \hat{\boldsymbol{q}}_{02} = \hat{\boldsymbol{q}}_{03} \times \hat{\boldsymbol{q}}_{01}. \tag{7.2}$$

In parallel with equation (3.11), the components of $\hat{\boldsymbol{Q}}_0$ relative to q are

$$\hat{Q}_{0ij} = \hat{w}_0 \delta_{ij} - (\hat{w}_0 - \hat{w}_{03}) \delta_{i3} \delta_{j3}, \tag{7.3}$$

and we note for future use the relation

$$\boldsymbol{b}_0 \cdot \hat{\boldsymbol{q}}_{02} = |\boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03}|, \tag{7.4}$$

implied by (7.2).

(*b*) *Perturbation of a model eigenvalue problem*

Our aim is to find wave normals close to $\boldsymbol{n}_0$ for which the modal roots w_1, w_2, each tending to $\hat{w}_0$ as $\chi \to 0$, $|\Delta \boldsymbol{n}| \to 0$, are degenerate for $\chi \ll 1$. We are thus concerned with the two small quantities χ and $|\Delta \boldsymbol{n}|$. However, a rigorous perturbation procedure can be formulated only in terms of a single small parameter (Kato 1982, ch. 2) and the relative magnitudes of χ and $|\Delta \boldsymbol{n}|$ depend on structural aspects, detailed in § 7 *e* below, of the degeneracy in the unperturbed state.

An appropriate formalism for the case in which the unperturbed modal roots are degenerate can be based on the model eigenvalue problem

$$Q_{ip}(\varepsilon) U_{\alpha p} = w_\alpha(\varepsilon) U_{\alpha i} \tag{7.5}$$

in which ε is a small parameter (not to be confused with the thermoelastic coupling constant defined by (6.3)). The components of the symmetric tensor $\boldsymbol{Q}(\varepsilon)$ and the eigenvectors $\boldsymbol{U}_\alpha$ relate to q and Latin and Greek subscripts take the values 1, 2, 3 and 1, 2, respectively. Substituting into (7.5) the perturbation expansions

$$Q_{ij}(\varepsilon) = \hat{Q}_{0ij} + \varepsilon Q^{(1)}_{ij} + \varepsilon^2 Q^{(2)}_{ij} + \ldots, \quad w_\alpha(\varepsilon) = \hat{w}_0 + \varepsilon w^{(1)}_\alpha + \varepsilon^2 w^{(2)}_\alpha + \ldots, \tag{7.6}$$

and incorporating the representation (7.3) of $\hat{Q}_{0ij}$, we obtain

$$-(\hat{w}_0 - \hat{w}_{03})U_{\alpha 3}\delta_{i3} + \{\varepsilon Q^{(1)}_{ip} + \varepsilon^2 Q^{(2)}_{ip} + \ldots\}U_{\alpha p} = \{\varepsilon w^{(1)}_\alpha + \varepsilon^2 w^{(2)}_\alpha + \ldots\}U_{\alpha i}. \quad (7.7)$$

A rationale for the retention of second-order terms emerges only towards the end of § 7 *e*.

When $i = \beta$, (7.7) yields

$$\{\varepsilon Q^{(1)}_{\beta p} + \varepsilon^2 Q^{(2)}_{\beta p} + \ldots\}U_{\alpha p} = \{\varepsilon w^{(1)}_\alpha + \varepsilon^2 w^{(2)}_\alpha + \ldots\}U_{\alpha\beta}, \quad (7.8)$$

and when $i = 3$,

$$\{\varepsilon Q^{(1)}_{p3} + \varepsilon^2 Q^{(2)}_{p3} + \ldots\}U_{\alpha p} = \{\hat{w}_0 - \hat{w}_{03} + \varepsilon w^{(1)}_\alpha + \varepsilon^2 w^{(2)}_\alpha + \ldots\}U_{\alpha 3}. \quad (7.9)$$

Equation (7.9) reveals that when $U_{\alpha 1}$ and $U_{\alpha 2}$ are taken to be of order unity, $U_{\alpha 3}$ is of order ε:

$$U_{\alpha 3} = \varepsilon(\hat{w}_0 - \hat{w}_{03})^{-1} Q^{(1)}_{\sigma 3} U_{\alpha\sigma} + \ldots, \quad (7.10)$$

with summation implied by the repeated subscript σ. (Note, however, that there is no summation on α in the right-hand sides of (7.7)–(7.9), and in similar expressions appearing below.) The result of introducing (7.10) into equation (7.8) is the two-dimensional eigenvalue problem

$$M_{\beta\sigma} U_{\alpha\sigma} = (\Delta w_\alpha) U_{\alpha\beta}, \quad (7.11)$$

where

$$\left.\begin{aligned} M_{\beta\gamma} &= \varepsilon Q^{(1)}_{\beta\gamma} + \varepsilon^2\{Q^{(2)}_{\beta\gamma} + (\hat{w}_0 - \hat{w}_{03})^{-1} Q^{(1)}_{\beta 3} Q^{(1)}_{\gamma 3}\} + \ldots, \\ \Delta w_\alpha &= \varepsilon w^{(1)}_\alpha + \varepsilon^2 w^{(2)}_\alpha + \ldots. \end{aligned}\right\} \quad (7.12)$$

(*c*) *The perturbed propagation condition*

When χ and $|\Delta \boldsymbol{n}|$ are small, the propagation condition obtained by combining equations (2.24) and $(2.25)_2$ is approximated by

$$Q_{ip} U_{\alpha p} = w_\alpha U_{\alpha i}, \quad (7.13)$$

where

$$Q_{ij} = \hat{Q}_{0ij} + \Delta\hat{Q}_{ij} - \mathrm{i}\chi \hat{w}_0^{-1} b_{0i} b_{0j}, \qquad w_\alpha = \hat{w}_0 + \Delta w_\alpha. \quad (7.14)$$

The components are referred to q and $\Delta\hat{\boldsymbol{Q}}$ is the perturbation of $\hat{\boldsymbol{Q}}_0$ due to the change of wave normal from $\boldsymbol{n}_0$ to $\boldsymbol{n}$. The steps leading from the model problem (7.5) to equation (7.11) reduce the perturbed propagation condition (7.13), (7.14) to the two-dimensional form,

$$(M_{\beta\sigma} - \mathrm{i}\chi \hat{w}_0^{-1} b_{0\beta} b_{0\sigma}) U_{\alpha\sigma} = (\Delta w_\alpha) U_{\alpha\beta}, \quad (7.15)$$

and we infer from (7.10) that, in the leading approximation, the displacement amplitudes $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ of the perturbed modes are coplanar with the displacement of the circularly polarized wave which, in the unperturbed state, propagates along the acoustic axis defined by $\boldsymbol{n}_0$.

From (7.2) and (7.4),

$$b_{0\alpha} = \boldsymbol{b}_0 \cdot \hat{\boldsymbol{q}}_{0\alpha} = |\boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03}| \delta_{\alpha 2}. \quad (7.16)$$

We can therefore rewrite (7.15) as

$$(M_{\beta\sigma} - 2\mathrm{i}\xi \hat{w}_0 \delta_{\beta 2}\delta_{\sigma 2}) U_{\alpha\sigma} = (\Delta w_\alpha) U_{\alpha\beta}, \quad (7.17)$$

with

$$\xi = \tfrac{1}{2}\chi \hat{w}_0^{-2} |\boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03}|^2. \tag{7.18}$$

The eigenvalues of the problem (7.17) are

$$\Delta w_\alpha = \tfrac{1}{2}[M_{11} + M_{22} - 2\mathrm{i}\xi\hat{w}_0 \pm \{(M_{11} - M_{22} + 2\mathrm{i}\xi\hat{w}_0)^2 + 4M_{12}^2\}^{1/2}].$$

They are equal if and only if

$$M_{11} - M_{22} = 0, \qquad M_{12} = \pm\xi\hat{w}_0, \tag{7.19}$$

and then

$$\Delta w_1 = \Delta w_2 = \Delta w_d = M_{11} - \mathrm{i}\xi\hat{w}_0. \tag{7.20}$$

For a given (small) value of χ, equations (7.19) and (7.18) determine the wave normals $\boldsymbol{n}_d^{(\pm)}$ for which the modes with displacement amplitudes $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ are degenerate. The signs in $\boldsymbol{n}_d^{(\pm)}$ correspond to those in $(7.19)_2$ which imply a two-fold set of solutions. The fact that (7.19) consists of two conditions suggests that, at the given value of χ, these degeneracies normally occur in isolated directions and not along closed curves surrounding the unperturbed wave normal $\boldsymbol{n}_0$.

From (7.19) and $(7.20)_3$,

$$N_{\beta\gamma} - (\Delta w_d)\delta_{\beta\gamma} = \begin{cases} \xi\hat{w}_0\pi_{\beta\gamma} & \text{when } \boldsymbol{n} = \boldsymbol{n}_d^{(+)}, \\ -\xi\hat{w}_0\bar{\pi}_{\beta\gamma} & \text{when } \boldsymbol{n} = \boldsymbol{n}_d^{(-)}, \end{cases} \tag{7.21}$$

where

$$N_{\beta\gamma} = M_{\beta\gamma} - 2\mathrm{i}\xi\hat{w}_0\delta_{\beta 2}\delta_{\gamma 2}, \qquad [\pi_{\beta\gamma}] = \pi = \begin{bmatrix} \mathrm{i} & 1 \\ 1 & -\mathrm{i} \end{bmatrix}, \tag{7.22}$$

and, as in § 3 *a*, an overbar denotes a complex conjugate. It follows that the eigenspace associated with the repeated eigenvalue Δw_d is the one-dimensional span of the isotropic vector

$$a = [1, \mathrm{i}]^{\mathrm{T}} \tag{7.23}$$

when $\boldsymbol{n} = \boldsymbol{n}_d^{(-)}$ and of $\bar{a}$ when $\boldsymbol{n} = \boldsymbol{n}_d^{(+)}$. The degeneracy is accordingly non-semisimple.

In the exact theory presented in §§ 2–6, the modal roots and displacement amplitudes are not eigenvalues and eigenvectors of $\boldsymbol{T}(w)$. Consequently, a degeneracy of a modal root w_k has to be distinguished from a degeneracy of the zero eigenvalue of $\boldsymbol{T}(w_k)$, the only exception arising when the wave normal is directed along a G-modal axis. We have found in this subsection that the incompatibility between a degeneracy of a modal root w_α and a non-semisimple degeneracy of the zero eigenvalue of $\boldsymbol{T}(w_\alpha)$, established in § 5 *d*, disappears at the level of approximation represented by equation $(7.14)_1$.

(*d*) *Determination of* $M_{\beta\gamma}$

The calculation of the matrix elements $M_{\beta\gamma}$ on which the solutions of equations (7.19) depend is carried out in this subsection under the simplifying assumption that the pre-stress $\bar{\boldsymbol{\sigma}}$ in the equilibrium configuration B_{e} is zero. The components of the isentropic elastic modulus then have the symmetry property

$$\hat{B}_{jikl} = \hat{B}_{ijkl}, \tag{7.24}$$

in addition to $(2.2)_2$ (Chadwick & Ogden 1971, § 2c)). Let $\boldsymbol{f}$, $\boldsymbol{g}$ be arbitrary vectors with components f_i, g_i relative to the arbitrary orthonormal basis e. We denote by $\boldsymbol{f}\hat{\boldsymbol{B}}\boldsymbol{g}$ the second-order tensor with components $B_{pirj}f_p g_r$. If $\boldsymbol{l}$, $\boldsymbol{m}$ are two more arbitrary vectors, equations (7.24) and $(2.2)_2$ validate the identity

$$\boldsymbol{l}\cdot\{(\boldsymbol{f}\hat{\boldsymbol{B}}\boldsymbol{g})\boldsymbol{m}\} = \boldsymbol{f}\cdot\{(\boldsymbol{l}\hat{\boldsymbol{B}}\boldsymbol{m})\boldsymbol{g}\}. \tag{7.25}$$

Replacing γ by γ_0 and making explicit the dependence on the wave normal $\boldsymbol{n}$, we can put the definition $(2.6)_2$ into direct notation as

$$\hat{\boldsymbol{Q}}(\boldsymbol{n}) = \gamma_0^{-1}\boldsymbol{n}\hat{\boldsymbol{B}}\boldsymbol{n}. \tag{7.26}$$

The first two terms on the right-hand side of equation $(7.14)_1$ are the components relative the special basis q of the tensors $\hat{\boldsymbol{Q}}_0 = \hat{\boldsymbol{Q}}(\boldsymbol{n}_0)$ and $\Delta\hat{\boldsymbol{Q}}(\boldsymbol{n})$, the latter being the perturbation of the former due to the change $\Delta\boldsymbol{n}$ in the wave normal. Thus, from (7.1) and (7.26),

$$\Delta\hat{\boldsymbol{Q}}(\boldsymbol{n}) = \hat{\boldsymbol{Q}}(\boldsymbol{n}) - \hat{\boldsymbol{Q}}(\boldsymbol{n}_0) = \gamma_0^{-1}\{(\boldsymbol{n}_0+\Delta\boldsymbol{n})\hat{\boldsymbol{B}}(\boldsymbol{n}_0+\Delta\boldsymbol{n}) - \boldsymbol{n}_0\hat{\boldsymbol{B}}\boldsymbol{n}_0\}. \tag{7.27}$$

Resolving $\boldsymbol{n}$ into components along and perpendicular to $\boldsymbol{n}_0$, we can rewrite equation (7.1) as

$$\Delta\boldsymbol{n} = \sin\vartheta\,\boldsymbol{m}_0 - (1-\cos\vartheta)\boldsymbol{n}_0, \qquad |\boldsymbol{m}_0| = 1, \qquad \boldsymbol{m}_0\cdot\boldsymbol{n}_0 = 0.$$

Expanding in powers of ϑ, we find that

$$\Delta\boldsymbol{n} = \delta\boldsymbol{n} + \tfrac{1}{2}\delta^2\boldsymbol{n} + \ldots, \tag{7.28}$$

with $\delta\boldsymbol{n} = \vartheta\boldsymbol{m}_0$, $\delta^2\boldsymbol{n} = -\vartheta^2\boldsymbol{n}_0$ and hence

$$\boldsymbol{n}_0\cdot\delta\boldsymbol{n} = 0, \qquad \delta^2\boldsymbol{n} = -|\delta\boldsymbol{n}|^2\boldsymbol{n}_0. \tag{7.29}$$

Equations (7.27), (7.28) and $(7.29)_2$ combine to give

$$\Delta\hat{\boldsymbol{Q}}(\boldsymbol{n}) = \gamma_0^{-1}\{\boldsymbol{n}_0\hat{\boldsymbol{B}}(\delta\boldsymbol{n}) + (\delta\boldsymbol{n})\hat{\boldsymbol{B}}\boldsymbol{n}_0 + (\delta\boldsymbol{n})\hat{\boldsymbol{B}}(\delta\boldsymbol{n}) - |\delta\boldsymbol{n}|^2\boldsymbol{n}_0\hat{\boldsymbol{B}}\boldsymbol{n}_0 + \ldots\}. \tag{7.30}$$

Comparing the expansions $(7.6)_1$ and (7.30), we make the identifications

$$\left.\begin{aligned}\varepsilon\hat{Q}_{ij}^{(1)} &= \gamma_0^{-1}\hat{\boldsymbol{q}}_{0i}\cdot[\{\boldsymbol{n}_0\hat{\boldsymbol{B}}(\delta\boldsymbol{n}) + (\delta\boldsymbol{n})\hat{\boldsymbol{B}}\boldsymbol{n}_0\}\hat{\boldsymbol{q}}_{0j}],\\ \varepsilon^2\hat{Q}_{ij}^{(2)} &= \gamma_0^{-1}\hat{\boldsymbol{q}}_{0i}\cdot[\{(\delta\boldsymbol{n})\hat{\boldsymbol{B}}(\delta\boldsymbol{n}) - |\delta\boldsymbol{n}|^2\boldsymbol{n}_0\hat{\boldsymbol{B}}\boldsymbol{n}_0\}\hat{\boldsymbol{q}}_{0j}].\end{aligned}\right\} \tag{7.31}$$

By means of the definitions

$$\boldsymbol{S}_{ij} = \gamma_0^{-1}(\hat{\boldsymbol{q}}_{0i}\hat{\boldsymbol{B}}\hat{\boldsymbol{q}}_{0j} + \hat{\boldsymbol{q}}_{0j}\hat{\boldsymbol{B}}\hat{\boldsymbol{q}}_{0i}), \qquad \boldsymbol{s}_{ij} = \boldsymbol{S}_{ij}\boldsymbol{n}_0, \tag{7.32}$$

and the identity (7.25), equations (7.31) can be expressed concisely as

$$\varepsilon\hat{Q}_{ij}^{(1)} = \boldsymbol{s}_{ij}\cdot(\delta\boldsymbol{n}), \qquad \varepsilon^2\hat{Q}_{ij}^{(2)} = \tfrac{1}{2}[(\delta\boldsymbol{n})\cdot\{\boldsymbol{S}_{ij}(\delta\boldsymbol{n})\} - |\delta\boldsymbol{n}|^2\boldsymbol{s}_{ij}\cdot\boldsymbol{n}_0]. \tag{7.33}$$

It now follows from equation $(7.12)_1$ that, to the second order in $|\delta\boldsymbol{n}|$,

$$M_{11} - M_{22} = 2\boldsymbol{p}\cdot(\delta\boldsymbol{n}) + (\delta\boldsymbol{n})\cdot\{\boldsymbol{F}(\delta\boldsymbol{n})\}, \quad M_{12} = \boldsymbol{q}\cdot(\delta\boldsymbol{n}) + \tfrac{1}{2}(\delta\boldsymbol{n})\cdot\{\boldsymbol{G}(\delta\boldsymbol{n})\}, \tag{7.34}$$

where

$$\boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{s}_{11} - \boldsymbol{s}_{22}), \qquad \boldsymbol{q} = \boldsymbol{s}_{12}, \tag{7.35}$$

$$\left.\begin{aligned}\boldsymbol{F} &= \tfrac{1}{2}(\boldsymbol{S}_{11} - \boldsymbol{S}_{22}) + (\hat{w}_0 - \hat{w}_{03})^{-1}(\boldsymbol{s}_{13}\otimes\boldsymbol{s}_{13} - \boldsymbol{s}_{23}\otimes\boldsymbol{s}_{23}),\\ \boldsymbol{G} &= \boldsymbol{S}_{12} + (\hat{w}_0 - \hat{w}_{03})^{-1}(\boldsymbol{s}_{13}\otimes\boldsymbol{s}_{23} + \boldsymbol{s}_{23}\otimes\boldsymbol{s}_{13}),\end{aligned}\right\} \tag{7.36}$$

$\otimes$ denoting a tensor product of vectors. Each of the tensors $\boldsymbol{S}_{ij}$, $\boldsymbol{F}$, $\boldsymbol{G}$ is symmetric. In connexion with (7.34), we note that equations (7.32), (7.25), (7.26) and (7.3) give

$$\tfrac{1}{2}(\boldsymbol{s}_{11} - \boldsymbol{s}_{22}) \cdot \boldsymbol{n}_0 = \hat{\boldsymbol{q}}_{01} \cdot \{\hat{\boldsymbol{Q}}(\boldsymbol{n}_0)\hat{\boldsymbol{q}}_{01}\} - \hat{\boldsymbol{q}}_{02} \cdot \{\hat{\boldsymbol{Q}}(\boldsymbol{n}_0)\hat{\boldsymbol{q}}_{02}\} = 0,$$

$$\tfrac{1}{2}\boldsymbol{s}_{12} \cdot \boldsymbol{n}_0 = \hat{\boldsymbol{q}}_{01} \cdot \{\hat{\boldsymbol{Q}}(\boldsymbol{n}_0)\hat{\boldsymbol{q}}_{02}\} = 0,$$

whence, from (7.35),

$$\boldsymbol{p} \cdot \boldsymbol{n}_0 = 0, \qquad \boldsymbol{q} \cdot \boldsymbol{n}_0 = 0. \tag{7.37}$$

By similar reasoning, $\boldsymbol{s}_{\alpha 3} \cdot \boldsymbol{n}_0 = 0$ and then, from (7.36), $(7.32)_2$ and (7.35),

$$\boldsymbol{F}\boldsymbol{n}_0 = \boldsymbol{p}, \qquad \boldsymbol{G}\boldsymbol{n}_0 = \boldsymbol{q}. \tag{7.38}$$

(*e*) *Types of acoustic axes*

The characteristic equation of $\hat{\boldsymbol{Q}}(\boldsymbol{n})$ specifies a three-sheeted slowness surface $\mathcal{S}$, centro-symmetric about a point O (see, for example, Boulanger & Hayes 1993, § 10.5). Two of the sheets of $\mathcal{S}$ intersect on the acoustic axis defined by $\boldsymbol{n}_0$ and it is evident from equation $(7.27)_1$ that $\Delta\hat{\boldsymbol{Q}}(\boldsymbol{n})$ describes the form of $\mathcal{S}$ in a neighbourhood of this axis. A detailed study of acoustic axes in classical elastodynamics has been made by Al'shits *et al.* (1985) and their paper should be consulted for further details of results quoted in the following discussion.

Any point P on a single sheet of $\mathcal{S}$ represents a plane harmonic wave with wave normal

$$\boldsymbol{n} = (\mathrm{OP})^{-1}\, \overrightarrow{\mathrm{OP}}$$

and group velocity codirectional with the normal to $\mathcal{S}$ at P (Boulanger & Hayes 1993, § 10.5). When $\boldsymbol{n} = \boldsymbol{n}_0$, P lies on two sheets of $\mathcal{S}$ and at nearby points on these sheets the displacement amplitudes of the associated plane harmonic waves are, to a first approximation, in the directions of the unit vectors

$$\cos\theta\hat{\boldsymbol{q}}_{01} + \sin\theta\hat{\boldsymbol{q}}_{02}, \qquad -\sin\theta\hat{\boldsymbol{q}}_{01} + \cos\theta\hat{\boldsymbol{q}}_{02},$$

the angle θ depending on $\Delta\boldsymbol{n}$. The group velocities of the two waves are $\frac{1}{2}(\bar{\rho}\hat{\omega}_0/\gamma_0)^{-1/2}$ times

$$\tfrac{1}{2}(\boldsymbol{s}_{11} + \boldsymbol{s}_{22}) \pm (\cos 2\theta\boldsymbol{p} + \sin 2\theta\boldsymbol{q}), \tag{7.39}$$

the vectors involved being defined by $(7.32)_2$ and (7.35). As $\Delta\boldsymbol{n}$ varies, the group velocity vectors delineate the normals to the intersecting sheets of $\mathcal{S}$ and hence the forms of the sheets themselves.

This local geometry clearly depends on $\boldsymbol{p}$ and $\boldsymbol{q}$, and three possibilities arise.

(i) When $\boldsymbol{p}$ and $\boldsymbol{q}$ are linearly independent the vectors (7.39) are the generators of a cone with elliptical base which, on account of (7.37), is in a plane orthogonal to $\boldsymbol{n}_0$. The acoustic axis is consequently said to be *of conical type*. Equations (7.34) give

$$M_{11} - M_{22} = 2\boldsymbol{p} \cdot (\delta\boldsymbol{n}), \qquad M_{12} = \boldsymbol{q} \cdot (\delta\boldsymbol{n}), \tag{7.40}$$

in the leading approximation, and the natural inference from (7.19) and (7.18) is that $|\delta\boldsymbol{n}|$ is of the same order of magnitude as χ.

(ii) When $\boldsymbol{p}$ and $\boldsymbol{q}$ are both zero, (7.39) provides a single normal. The intersecting sheets of $\mathcal{S}$ then touch on the acoustic axis which is said to be *of tangential type*. The leading approximation supplied by equations (7.34) is

$$M_{11} - M_{22} = (\delta\boldsymbol{n}) \cdot \{\boldsymbol{F}(\delta\boldsymbol{n})\}, \qquad M_{12} = \tfrac{1}{2}(\delta\boldsymbol{n}) \cdot \{\boldsymbol{G}(\delta\boldsymbol{n})\}, \tag{7.41}$$

and the implication of (7.19) and (7.18) is that $|\delta\boldsymbol{n}|$ is of the same order of magnitude as $\chi^{1/2}$. The necessity of including second-order terms from equations (7.6) onwards is now apparent.

(iii) When $\boldsymbol{p}$ and $\boldsymbol{q}$ are linearly dependent but not both zero, the vectors (7.39) are confined to a plane orthogonal to $(\boldsymbol{s}_{11} + \boldsymbol{s}_{22}) \times \boldsymbol{p}$. The acoustic axis as said to be *of ridge* (or *wedge*) *type*. Commonly (and specifically in transversely isotropic media (Chadwick 1989, § 5(b)), acoustic axes of this type occur along a curve and are then created by a crossover of two sheets of $\mathcal{S}$. The quadratic terms in equations (7.34) cannot be dropped unconditionally, as in case (i), since the linear terms are of comparable smallness when $\delta\boldsymbol{n}$ is sufficiently close to a direction orthogonal to $\boldsymbol{p}$.

When the relative magnitudes of χ and $|\delta\boldsymbol{n}|$ are known, it is possible to form a composite parameter in terms of which the perturbation procedure can be made precise. The domain of validity of the procedure is also specified. For acoustic axes of conical and tangential types it consists of wave normals deviating from $\boldsymbol{n}_0$ by angles of order χ and $\chi^{1/2}$, respectively. For an acoustic axis of ridge type the angle of deviation varies, being of order $\chi^{1/2}$ when $\boldsymbol{n} - \boldsymbol{n}_0$ is in or near to one of the directions orthogonal to $\boldsymbol{p}$ and of order χ when $(\boldsymbol{n} - \boldsymbol{n}_0) \cdot \boldsymbol{p} \gg \chi^{1/2}$.

(*f*) *Solutions of equations (7.19)*

(i) When the acoustic axis is of conical type, equations (7.40) apply and (7.19) become

$$\boldsymbol{p} \cdot (\delta\boldsymbol{n}) = 0, \qquad \boldsymbol{q} \cdot (\delta\boldsymbol{n}) = \pm\xi\hat{w}_0. \tag{7.42}$$

By $(7.29)_1$ and (7.37), $\delta\boldsymbol{n}$, $\boldsymbol{p}$, $\boldsymbol{q}$ are all orthogonal to $\boldsymbol{n}_0$, so there exist scalars α, β such that $\delta\boldsymbol{n} = \alpha\boldsymbol{p} + \beta\boldsymbol{q}$. Entering this expression into equations (7.42) and solving for α, β, we conclude that the wave normals for which degeneracy occurs are $\boldsymbol{n}_d^{(\pm)} = \boldsymbol{n}_0 + \Delta\boldsymbol{n}_d^{(\pm)}$, where

$$\Delta\boldsymbol{n}_d^{(\pm)} = \mp\xi\hat{w}_0|\boldsymbol{p} \times \boldsymbol{q}|^{-2}\boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{q}). \tag{7.43}$$

It can easily be checked from (7.43) and (7.18) that

$$\arccos(\boldsymbol{n}_d^{(+)} \cdot \boldsymbol{n}_d^{(-)}) = 2^{-1/2}\chi\hat{w}_0^{-1}|\boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03}|^2|\boldsymbol{p}||\boldsymbol{p} \times \boldsymbol{q}|^{-1}.$$

The degeneracies are therefore very close together.

(ii) When the acoustic axis is of tangential type, equations (7.41) and (7.19) combine to give

$$(\delta\boldsymbol{n}) \cdot \{\boldsymbol{F}(\delta\boldsymbol{n})\} = 0, \qquad (\delta\boldsymbol{n}) \cdot \{\boldsymbol{G}(\delta\boldsymbol{n})\} = \pm 2\xi\hat{w}_0. \tag{7.44}$$

Since $\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{0}$, equations (7.38) imply that $\boldsymbol{F}$ and $\boldsymbol{G}$ each have a zero eigenvalue. Let the other (real) eigenvalues of $\boldsymbol{F}$ and $\boldsymbol{G}$ be λ_1, λ_2, and μ_1, μ_2, respectively, with associated (orthogonal) unit eigenvectors $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ and $\cos\psi\boldsymbol{f}_1 + \sin\psi\boldsymbol{f}_2, -\sin\psi\boldsymbol{f}_1 + \cos\psi\boldsymbol{f}_2$. Then, if

$$\mathsf{x} = \boldsymbol{f}_1 \cdot (\delta\boldsymbol{n}), \qquad \mathsf{y} = \boldsymbol{f}_2 \cdot (\delta\boldsymbol{n}),$$

equations (7.44) can be rewritten as

$$\left.\begin{aligned} \lambda_1\mathsf{x}^2 + \lambda_2\mathsf{y}^2 = 0, \\ \mu_1(\mathsf{x}\cos\psi + \mathsf{y}\sin\psi)^2 + \mu_2(-\mathsf{x}\sin\psi + \mathsf{y}\cos\psi)^2 = \pm 2\xi\hat{w}_0. \end{aligned}\right\} \tag{7.45}$$

Evidently, x and y can be interpreted as rectangular Cartesian coordinates in a plane orthogonal to $\boldsymbol{n}_0$ with origin O located at the tip of the vector $\boldsymbol{n}_0$.

Table 1. *Possible numbers of real solutions of equations (7.45) when* $|\lambda_1| + |\lambda_2| > 0$

	$\lambda_1\lambda_2 = 0$	$\lambda_1\lambda_2 < 0$
$\mu_1\mu_2 = 0$	(1,1)	(1,1) or (2,2)
$\mu_1\mu_2 < 0$	(2,0) or (0,2)	(2,0), (0,2), (2,2), (4,0) or (0,4)
$\mu_1\mu_2 > 0$	(2,0) or (0,2)	(4,0) or (0,4)

Equations (7.45) have real solutions only if

$$\lambda_1\lambda_2 \leqslant 0, \qquad |\mu_1| + |\mu_2| > 0.$$

The nature of the solutions is determined by λ_1, λ_2, μ_1, μ_2 and ψ, and we see from equations (7.36) and (7.32) that these quantities relate exclusively to the unperturbed state. Characterizing the solutions of (7.45) is therefore tantamount to a sub-classification of acoustic axes of tangential type.

When equations (7.45) have no solution there is no degeneracy near the acoustic axis and this underlying degeneracy is erased by a thermoelastic perturbation.

When λ_1 and λ_2 are both zero, equation $(7.45)_1$ is nugatory and the solutions correspond to the points of the curve represented by $(7.45)_2$. The curve is an ellipse with centre O when $\mu_1\mu_2 > 0$, a pair of parallel lines equidistant from O when $\mu_1\mu_2 = 0$, and two hyperbolas with common asymptotes and centre O when $\mu_1\mu_2 < 0$. Thus all the solutions (x, y) are close to O when $\mu_1\mu_2 > 0$, and when $\mu_1\mu_2 \leqslant 0$ a sufficiently large neighbourhood of O contains two arcs of solutions.

When $|\lambda_1| + |\lambda_2| > 0$, equation $(7.45)_1$ represents one or two lines through O according as $\lambda_1\lambda_2 = 0$ or $\lambda_1\lambda_2 < 0$. If the former case applies and $\mu_1\mu_2 \leqslant 0$, the single line may be parallel to the pair of lines or coincident with an asymptote of the hyperbolas represented by $(7.45)_2$ and no solution then exists. Otherwise, solutions occur in pairs, as specified in table 1, where (m, n) means that there are m solutions $\delta\boldsymbol{n}_d^{(+)}$ and n solutions $\delta\boldsymbol{n}_d^{(-)}$: in other words, there exist m solutions (x, y) of (7.45) with the upper sign in $(7.45)_2$ and n solutions with the lower sign. The overall conclusion is that the underlying degeneracy either disappears or divides into two or four degeneracies when subjected to a thermoelastic perturbation.

Recalling the definition (7.18), we gather from $(7.45)_2$ that, for each of the possibilities set out in table 1, $|\delta\boldsymbol{n}_d^{(\pm)}|$ is a multiple of $\chi^{1/2}$. Pairs of degeneracies are therefore close together, but more widely separated than in case (i).

(iii) When the acoustic axis is of ridge type, equations (7.34) hold and $\boldsymbol{q}$ is a scalar multiple of $\boldsymbol{p} : \boldsymbol{q} = \eta\boldsymbol{p}$ say. Equations (7.19) accordingly take the form

$$\left.\begin{aligned} 2\boldsymbol{p}\cdot(\delta\boldsymbol{n}) + (\delta\boldsymbol{n})\cdot\{\boldsymbol{F}(\delta\boldsymbol{n})\} &= 0,\\ 2\eta\boldsymbol{p}\cdot(\delta\boldsymbol{n}) + (\delta\boldsymbol{n})\cdot\{\boldsymbol{G}(\delta\boldsymbol{n})\} &= \pm 2\xi\hat{w}_0. \end{aligned}\right\} \tag{7.46}$$

Since $\delta\boldsymbol{n}$, $\boldsymbol{p}$, $\boldsymbol{p}\times\boldsymbol{n}_0$ are all orthogonal to $\boldsymbol{n}_0$ (see $(7.29)_1$ and $(7.37)_1$), we can set

$$\delta\boldsymbol{n} = |\boldsymbol{p}|^{-1}(x\boldsymbol{p} + y\boldsymbol{p}\times\boldsymbol{n}_0) \tag{7.47}$$

and regard (x, y) as rectangular Cartesian coordinates, as in case (ii). With the introduction of (7.47), equations (7.46) become

$$\left.\begin{aligned} 2|\boldsymbol{p}|x + F_{11}x^2 + 2F_{12}xy + F_{22}y^2 = 0, \\ 2\eta|\boldsymbol{p}|x + G_{11}x^2 + 2G_{12}xy + G_{22}y^2 = \pm 2\xi\hat{w}_0, \end{aligned}\right\} \tag{7.48}$$

where

$$\begin{aligned} F_{11} &= |\boldsymbol{p}|^{-2}\boldsymbol{p}\cdot(\boldsymbol{F}\boldsymbol{p}), \\ F_{12} &= |\boldsymbol{p}|^{-2}(\boldsymbol{p}\times\boldsymbol{n}_0)\cdot(\boldsymbol{F}\boldsymbol{p}), \\ F_{22} &= |\boldsymbol{p}|^{-2}(\boldsymbol{p}\times\boldsymbol{n}_0)\cdot\{\boldsymbol{F}(\boldsymbol{p}\times\boldsymbol{n}_0)\}, \end{aligned}$$

and the definitions of G_{11}, G_{12}, G_{22} are exactly similar.

We assume initially that $F_{22} \neq 0$, $G_{22} \neq 0$. The curves C_1, C_2 represented by equations $(7.48)_{1,2}$ are conics, C_1 passing through the origin O and, due to the smallness of ξ, C_2 approaching closely to O. C_1 is tangential to the y-axis at O and either the whole of C_1 or the branch containing O lies in the half-plane $x \geqslant 0$ or $x \leqslant 0$ according as $F_{22} \lesseqgtr 0$. C_2 intersects the y-axis at $(0, \pm\{2\xi\hat{w}_0|G_{22}|^{-1}\}^{1/2})$, both points being associated with the upper sign in $(7.48)_2$ when $G_{22} > 0$ and with the lower sign when $G_{22} < 0$. C_2 also cuts the x-axis twice, the point of intersection nearest to O being, to the first order in ξ, $(\pm\{\eta|\boldsymbol{p}|\}^{-1}\xi\hat{w}_0, 0)$, with the signs corresponding to those in $(7.48)_2$.

All the above-mentioned properties of C_1 and C_2 are also possessed by the parabolas

$$y^2 = -2|\boldsymbol{p}|F_{22}^{-1}x \quad \text{and} \quad y^2 = 2\eta|\boldsymbol{p}|G_{22}^{-1}[\pm\{\eta|\boldsymbol{p}|\}^{-1}\xi\hat{w}_0 - x], \tag{7.49}$$

the signs in $(7.49)_2$ answering to those in $(7.48)_2$. Inspection of (7.48) confirms that the parabolas approximate C_1 and C_2 near O. Let

$$h = G_{22} - \eta F_{22}.$$

Then the parabolas (7.49) have a real intersection, consisting of two points equidistant from the x-axis, if $h > 0$ when the upper sign applies, and if $h < 0$ when the lower sign holds. Returning to (7.47), we thus obtain the solutions

$$\left.\begin{aligned} \boldsymbol{n}_d^{(+)} &= \boldsymbol{n}_0 - \{h|\boldsymbol{p}|^2\}^{-1}F_{22}\xi\hat{w}_0\boldsymbol{p} \pm |\boldsymbol{p}|^{-1}(2h^{-1}\xi\hat{w}_0)^{1/2}(\boldsymbol{p}\times\boldsymbol{n}_0) \quad \text{when} \quad h > 0, \\ \boldsymbol{n}_d^{(-)} &= \boldsymbol{n}_0 + \{h|\boldsymbol{p}|^2\}^{-1}F_{22}\xi\hat{w}_0\boldsymbol{p} \pm |\boldsymbol{p}|^{-1}(-2h^{-1}\xi\hat{w}_0)^{1/2}(\boldsymbol{p}\times\boldsymbol{n}_0) \quad \text{when} \quad h < 0, \end{aligned}\right\} \tag{7.50}$$

in the leading approximation.

The initial assumption can now be relaxed. Equations (7.50) remain valid when $F_{22} = 0$ (and $(7.49)_1$ reduces to the line $x = 0$) or $G_{22} = 0$ ($(7.49)_2$ then collapsing into the parallel lines $x = \pm\{\eta|\boldsymbol{p}|\}^{-1}\xi\hat{w}_0$). However, when $h = 0$ (in particular, if F_{22} and G_{22} are both zero), the parabolas (7.49) are disjoint and equations (7.48) have no solution near O. The conclusion, similar to that reached in case (ii), is that the effect of a thermoelastic perturbation is either to remove the underlying degeneracy or to split it into two.

Finally, we observe from (7.50) that the orders of magnitude of the components of $\delta\boldsymbol{n}_d^{(\pm)}$ in the directions of $\boldsymbol{p}$ and $\boldsymbol{p}\times\boldsymbol{n}_0$ are in full agreement with the remarks at the end of § 7 *e*.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

8. Properties of degenerate modes near an isentropic acoustic axis

(*a*) *The solution in the leading approximation*

We have considered in § 7 *c* two modes of thermoelastic wave propagation for which the dimensionless frequency χ is small and the wave normal $\boldsymbol{n}$ is close to a unit vector $\boldsymbol{n}_0$ defining an acoustic axis in the isentropic limit. In the unperturbed state, given by $\chi = 0$, $\boldsymbol{n} = \boldsymbol{n}_0$, the two modes have a common modal root $\hat{w}_0$. In the perturbed state the modal roots are $\hat{w}_0 + \Delta w_\alpha$, and when $\Delta w_1 \neq \Delta w_2$ the displacement amplitudes are constant, as in equation (2.19). As shown in § 7 *c*, a degeneracy, specified by equations $(7.20)_{1,2}$, is non-semisimple and the displacement amplitude of the composite degenerate mode then depends linearly on x, as in equation (4.13).

In the small perturbation approximation the non-degenerate behaviour of the two modes close to the underlying degeneracy is governed by the two-dimensional eigenvalue problem (7.17), (7.18), derived from (2.24) and $(2.25)_2$. At a degeneracy, the vectors $\boldsymbol{U}^{(1)}$, $\boldsymbol{U}^{(2)}$ appearing in the displacement amplitude satisfy equations $(4.20)_1$ and $(4.24)_1$ and we now require the approximate forms of these relations analogous to (7.17). It is convenient to revert to matrix notation and it is understood from now on that $U^{(1)}$, $U^{(2)}$, b_0 are the 2×1 column vectors of components of $\boldsymbol{U}^{(1)}$, $\boldsymbol{U}^{(2)}$, $\boldsymbol{b}_0$ relative to the orthonormal pair $\{\hat{\boldsymbol{q}}_{01}, \hat{\boldsymbol{q}}_{02}\}$ defined by (7.2). Comparing equations (2.24) and $(4.20)_1$, then combining (7.17) (the approximation to (2.24)) with $(7.22)_1$ and $(7.20)_{1,2}$, we obtain

$$NU^{(2)} = (\Delta w_d)U^{(2)}. \tag{8.1}$$

The corresponding reduction of equation $(4.24)_1$ is

$$NU^{(1)} = (\Delta w_d)U^{(1)} - 2\hat{w}_0 U^{(2)}, \tag{8.2}$$

w_d being replaced by $\hat{w}_0 + \Delta w_d$ and small terms on the right-hand side neglected. This step may, at first sight, seem inconsistent with the retention in N of a term proportional to χ (see $(7.22)_1$ and (7.18)). However, $U^{(2)}$ is determined by equation (8.1) only to within a scalar multiplier and this arbitrariness is the means of balancing small terms in (8.2). On the basis of equation (7.11), $\boldsymbol{U}^{(1)} \cdot \hat{\boldsymbol{q}}_{03}$ and $\boldsymbol{U}^{(2)} \cdot \hat{\boldsymbol{q}}_{03}$ are negligible in the leading approximation represented by equations (8.1) and (8.2).

In conformity with the non-semisimple nature of the degeneracy, $U^{(1)}$ is characterized by equation (8.2) as a generalized eigenvector of the 2×2 complex symmetric matrix N. On pre-multiplying (8.2) by $U^{(2)\mathrm{T}}$ and invoking (8.1), we have

$$U^{(2)\mathrm{T}} U^{(2)} = 0. \tag{8.3}$$

This may also be regarded as the first approximation to (4.21).

It has been noted in the penultimate paragraph of § 7 *c* that $\bar{a}$ or a, defined by (7.23), is an eigenvector of N associated with the repeated eigenvalue Δw_d according as the wave normal of the degenerate mode is $\boldsymbol{n}_d^{(+)}$ or $\boldsymbol{n}_d^{(-)}$. Using the superscripts (+) and (−) systematically to refer to these possibilities, we can therefore state the general solution of equation (8.1) as

$$U^{(2)(+)} = -\mathrm{i}\xi c_1^{(+)} \bar{a}, \qquad U^{(2)(-)} = -\mathrm{i}\xi c_1^{(-)} a, \tag{8.4}$$

$c_1^{(\pm)}$ being arbitrary complex constants with the physical dimension of length. The isotropy condition (8.3) is clearly satisfied by (8.4) and the inclusion of the factor ξ provides the degree of smallness looked for above.

Substituting from (8.4) and recalling (7.21), we deduce from (8.2) the relations

$$\pi U^{(1)(+)} = 2\mathrm{i}c_1^{(+)}\bar{a}, \qquad \bar{\pi}U^{(1)(-)} = -2\mathrm{i}c_1^{(-)}a. \tag{8.5}$$

From $(7.22)_2$ and (7.23),

$$\pi a = 2\mathrm{i}\bar{a}, \qquad \bar{\pi}\bar{a} = -2\mathrm{i}a.$$

A particular solution of equations (8.5) is hence

$$U^{(1)(+)} = c_1^{(+)}a, \qquad U^{(1)(-)} = c_1^{(-)}\bar{a},$$

and the general solution is

$$U^{(1)(+)} = c_1^{(+)}a + c_2^{(+)}\bar{a}, \qquad U^{(1)(-)} = c_1^{(-)}\bar{a} + c_2^{(-)}a, \tag{8.6}$$

$c_2^{(\pm)}$ being a second pair of arbitrary complex constants.

We conclude from equations (4.13), (8.4) and (8.6) that the projection on the plane orthogonal to $\hat{\boldsymbol{q}}_{03}$ of the displacement of the degenerate mode is given by

$$\left.\begin{aligned} u_d^{(+)} &= \exp\{\mathrm{i}\omega(s_d^{(+)}x - t)\}\{c_1^{(+)}(a + \xi\omega s_0 x\bar{a}) + c_2^{(+)}\bar{a}\}, \\ u_d^{(-)} &= \exp\{\mathrm{i}\omega(s_d^{(-)}x - t)\}\{c_1^{(-)}(\bar{a} + \xi\omega s_0 x a) + c_2^{(-)}a\}, \end{aligned}\right\} \tag{8.7}$$

where, from (4.14),

$$s_d^{(\pm)} = s_0(1 - \tfrac{1}{2}\hat{w}_0^{-1}\Delta w_d^{(\pm)}), \qquad s_0 = (\gamma_0\hat{w}_0/\bar{\rho})^{-1/2}. \tag{8.8}$$

Equations $(7.20)_3$, $(7.12)_1$ and $(7.33)_1$ yield

$$\Delta w_d^{(\pm)} = \boldsymbol{s}_{11}\cdot(\delta\boldsymbol{n}^{(\pm)}) - \mathrm{i}\xi\hat{w}_0,$$

in the leading approximation, and, from (7.32) and (7.26),

$$\boldsymbol{s}_{11} = 2\hat{\boldsymbol{Q}}(\hat{\boldsymbol{q}}_{01})\boldsymbol{n}_0.$$

Hence $(8.8)_1$ can be rewritten as

$$s_d^{(\pm)} = s_0\{1 - s_0\boldsymbol{g}\cdot(\delta\boldsymbol{n}^{(\pm)}) + \tfrac{1}{2}\mathrm{i}\xi\}, \tag{8.9}$$

where

$$\boldsymbol{g} = (\gamma_0 s_0/\bar{\rho})\hat{\boldsymbol{Q}}(\hat{\boldsymbol{q}}_{01})\boldsymbol{n}_0$$

is the group velocity, in the isentropic limit, of the homogeneous plane wave with wave normal $\boldsymbol{n}_0$ and displacement amplitude $\hat{\boldsymbol{q}}_{01}$ (Fedorov 1968, § 21). If the acoustic axis is of tangential type, $\hat{\boldsymbol{q}}_{01}$ can be replaced by any unit vector orthogonal to $\hat{\boldsymbol{q}}_{03}$.

The constants $c_1^{(\pm)}$ and $c_2^{(\pm)}$ in (8.7) may be specified by conditions on the plane $x = 0$. In particular, if

$$u_d^{(+)}|_{x=0} = c^{(+)}\bar{a}\exp(-\mathrm{i}\omega t), \qquad u_d^{(-)}|_{x=0} = c^{(-)}a\exp(-\mathrm{i}\omega t),$$

we have

$$c_1^{(\pm)} = 0, \qquad c_2^{(\pm)} = c^{(\pm)},$$

and each of (8.7) represents a circularly polarized wave. This is in keeping with the result stated in the final paragraph of § 4.

The scaled temperature changes $\phi_d^{(\pm)}$ accompanying the displacements (8.7) are supplied by equations (4.13), $(4.20)_2$ and $(4.24)_2$. Setting $w_d^{(\pm)} = \hat{w}_0 + \Delta w_d^{(\pm)}$, $\boldsymbol{b} = \boldsymbol{b}_0$ and discarding small terms, we secure the leading approximations

$$\Phi^{(2)} = -\hat{w}_0^{-1/2}b_0^{\mathrm{T}}U^{(2)}, \qquad \Phi^{(1)} = -\hat{w}_0^{-1/2}b_0^{\mathrm{T}}(U^{(1)} + U^{(2)}). \tag{8.10}$$

From (8.4) and (8.6), $U^{(2)\mathrm{T}}U^{(2)}$ is small compared with $U^{(1)\mathrm{T}}U^{(1)}$. Equations (4.13) and (8.10) therefore yield

$$\phi_d = -\hat{w}_0^{-1/2} \exp\{\mathrm{i}\omega(s_d x - t)\} b_0^{\mathrm{T}}(U^{(1)} + \mathrm{i}\omega s_0 x U^{(2)}),$$

and on substituting from (8.4) and (8.6) and making use of equations (7.23) and (7.16), we find that

$$\phi_d^{(\pm)} = \mp \mathrm{i}\hat{w}_0^{-1/2} |\boldsymbol{b}_0 \times \hat{\boldsymbol{q}}_{03}| \exp\{\mathrm{i}\omega(s_d^{(\pm)} x - t)\}\{c_1^{(\pm)}(1 - \xi\omega s_0 x) - c_2^{(\pm)}\}. \tag{8.11}$$

It may easily be verified from (8.7) and (8.11) that, at this level of approximation, the displacement and the scaled temperature change are related by

$$\phi_d^{(\pm)} = -\hat{w}_0^{-1/2} b_0^{\mathrm{T}} u_d^{(\pm)}.$$

(*b*) *Properties of the solution*

The wave structure displayed by equations (8.7) and (8.11), with coordinate dependent amplitudes, is encountered in other branches of wave propagation theory when non-semisimple degeneracy of the slowness (or wave number) occurs. Examples are electromagnetic waves travelling along a direction of degeneracy of the complex refractive index in optically absorbing crystals of low symmetry (Fedorov & Goncharenko 1963; Fedorov 1976) and degenerate surface waves in anisotropic elastic media (Lothe & Barnett 1976; Ting 1997).

The linear dependence of the amplitudes on x reduces the damping of the composite degenerate mode and the effect is especially pronounced when $\xi\omega s_0 x$ is small, that is when the exponential decay resulting from the factor $\exp(-\frac{1}{2}\xi\omega s_0 x)$ has not progressed far. As a measure of the intensity of the motion we can take the average kinetic energy

$$K = \frac{1}{4}\frac{\bar{\rho}\omega}{\pi}\int_{\tau}^{\tau+(2\pi/\omega)} \dot{u}^{\mathrm{T}}\dot{u}\,\mathrm{d}t \tag{8.12}$$

per unit volume in the equilibrium configuration B_{e}. Here u is the real or imaginary part of the displacement amplitude $\exp\{\mathrm{i}\omega(sx - t)\}A(x)$ and τ is arbitrary: $\bar{\rho}$, of course, is not a complex conjugate. Then, evaluating the integral in (8.12),

$$K = \tfrac{1}{4}\bar{\rho}\omega^2 A^{\mathrm{T}}(x)\bar{A}(x)$$

(cf. Boulanger & Hayes 1993, § 10.3). Reading off $A(x)$ from equations (8.7) and (8.9) and making use of (7.23), we find that

$$\begin{aligned} K_d^{(\pm)} = \tfrac{1}{2}\bar{\rho}\omega^2 \exp(-\xi\omega s_0 x)[c_1^{(\pm)}\bar{c}_1^{(\pm)}\{1 + (\xi\omega s_0 x)^2\} + c_2^{(\pm)}\bar{c}_2^{(\pm)} \\ + (c_1^{(\pm)}\bar{c}_2^{(\pm)} + \bar{c}_1^{(\pm)}c_2^{(\pm)})\xi\omega s_0 x]. \end{aligned} \tag{8.13}$$

When

$$b_0^{\mathrm{T}} u_d^{(\pm)}|_{x=0} = 0,$$

it follows from equations (8.7), (7.16) and (7.23) that $c_1^{(\pm)} = c_2^{(\pm)}$. Equation (8.13) then simplifies to

$$\begin{aligned} K_d^{(\pm)} &= \bar{\rho}\omega^2 c_1^{(\pm)}\bar{c}_1^{(\pm)} \exp(-\xi\omega s_0 x)\{1 + \xi\omega s_0 x + \tfrac{1}{2}(\xi\omega s_0 x)^2\} \\ &= \bar{\rho}\omega^2 c_1^{(\pm)}\bar{c}_1^{(\pm)}\{1 + O(\xi\omega s_0 x)^3\} \quad \text{as} \quad \xi\omega s_0 x \to 0. \end{aligned}$$

The damping is thus extremely weak for small values of $\xi\omega s_0 x$ in this case. When $u_d^{(\pm)}$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

is not orthogonal to b_0 at $x = 0$, $K_d^{(\pm)}$ depends on the orientation of the displacement amplitude at $x = 0$ and, by (8.13), the attenuation is much greater.

The definitions (7.18) and $(2.6)_4$ imply that $\xi\omega s_0 x$ is small for ultrasonic frequencies and distances of travel appropriate to crystal acoustics. This may not be so, however, for phonon processes in some solids. For frequencies and distances typical of seismic wave propagation, $\xi\omega s_0 x$ is not small and the exponential damping inherent in (8.7) and (8.11) becomes decisive. Generally, the possibility of $\xi\omega s_0 x$ changing from a small to a large number as ω or x increases permits an evolution of the displacement amplitudes in (8.7) similar to that found in the optical analogue by Fedorov & Goncharenko (1963) and Fedorov (1976). We see from (8.7) that the displacements $u_d^{(\pm)}$ are closely aligned with their values at $x = 0$ when $\xi\omega s_0 x \ll 1$, acquire linear polarization when

$$|c_1^{(\pm)}|^2(\xi\omega s_0 x)^2 + 2\,\mathrm{Re}(c_1^{(\pm)}\bar{c}_2^{(\pm)})\xi\omega s_0 x - |c_1^{(\pm)}|^2 + |c_2^{(\pm)}|^2 = 0,$$

and approach the circular polarization $\bar{a}$ (for $u_d^{(+)}$) or a (for $u_d^{(-)}$) when $\xi\omega s_0 x \gg 1$. It is possible in this way for one of the displacements (8.7) to reverse the handedness of a circular polarization induced at $x = 0$, passing through a state of linear polarization *en route*. It is apparent from (8.11) that these variations in the displacement are attended by a similarly complicated modulation of the scaled temperature change.

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Appendix A. Basic matrix relations

In terms of an arbitrary symmetric 3×3 matrix A, an arbitrary 3×1 column vector c and arbitrary non-zero scalars α, β, all complex-valued, let 4×4 matrices S, S' and 3×3 matrices B, T, T' be defined by

$$S = \begin{bmatrix} A & c \\ c^{\mathrm{T}} & \alpha \end{bmatrix}, \qquad S' = \begin{bmatrix} I & (2\beta)^{-1}c \\ (2\beta)^{-1}c^{\mathrm{T}} & 1 \end{bmatrix},$$

$$B = A - \beta^{-1}cc^{\mathrm{T}}, \qquad T = A - \alpha^{-1}cc^{\mathrm{T}}, \qquad T' = I - (\alpha^2\beta)^{-1}(\alpha - \beta)cc^{\mathrm{T}},$$

I being the 3×3 identity matrix. Then the following relations can be established by elementary methods, a star indicating the adjugate, i.e. the matrix of cofactors, det the determinant and tr the trace of a matrix.

$$\det S = \alpha \det A - c^{\mathrm{T}}A^*c = (\alpha - \beta)\det A + \beta \det B = \alpha \det T, \tag{A 1}$$

$$\det T = \det A - \alpha^{-1}c^{\mathrm{T}}T^*c, \tag{A 2}$$

$$S\begin{bmatrix} -A^*c \\ \det A \end{bmatrix} = \begin{bmatrix} 0 \\ \det S \end{bmatrix}, \tag{A 3}$$

$$S^* = \begin{bmatrix} \alpha T^* & -A^*c \\ -(A^*c)^{\mathrm{T}} & \det A \end{bmatrix}, \tag{A 4}$$

$$\mathrm{tr}\, S^* = \alpha\, \mathrm{tr}\, T^* + \det A = (\alpha - \beta)\,\mathrm{tr}\, A^* + \beta\, \mathrm{tr}\, B^* + \det A, \tag{A 5}$$

$$\begin{aligned} \mathrm{tr}(S^*S') &= \mathrm{tr}\, S^* - \beta^{-1}c^{\mathrm{T}}A^*c = \mathrm{tr}\, S^* + \beta^{-1}\det S - \alpha\beta^{-1}\det A \\ &= \alpha\, \mathrm{tr}\, T^* - \beta^{-1}\{(\alpha - \beta)\det A - \det S\}, \end{aligned} \tag{A 6}$$

$$T^* = \alpha^{-1}\{(\alpha - \beta)A^* + \beta B^*\}, \tag{A 7}$$

$$\operatorname{tr}(T^* T') = \alpha^{-1} \operatorname{tr}(S^* S') - \alpha^{-2} \det S. \tag{A 8}$$

Equations (A 1) to (A 8) are applied to the theory of thermoelastic waves by making the substitutions

$$A = wI - \tilde{Q}, \qquad c = w^{1/2} b, \qquad \alpha = w + \mathrm{i}\chi, \qquad \beta = w. \tag{A 9}$$

Then, with reference to equations (2.12), $(2.25)_1$ and (2.15), $S = S(w)$, $T = T(w)$ and

$$B = wI - (\tilde{Q} + bb^{\mathrm{T}}) = wI - \hat{Q}, \qquad S' = \frac{\mathrm{d}}{\mathrm{d}w} S(w), \qquad T' = \frac{\mathrm{d}}{\mathrm{d}w} T(w). \tag{A 10}$$

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